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An Essay in Combinatory Dynamic Logic*

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We propose a refinement of Kripke modal logic, and in particular of PDL.

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Preface

The theory developed here, although deriving its motivation and parts of its terminology from programming theory, can be viewed as a theory for reasoning about action in general; hence the term dynamic logic. (Harel, 1979)

Syntactically, modal logic in general, and dynamic logic in particular, lacks the "static" notion of possible world, or execution state, which is the essence of its semantics. We introduce the states in the syntax, together with a universe-action, connecting each pair of states, and adopt appropriate axioms; we add the adjective "combinatory" for such a revision of the modal logic.

The present paper is aimed at justifying our thesis: The combinatory dynamic/modal logic presents the natural state of affairs.

Chapter 0: Prologue

The present paper, though self-contained, is not designed as an introduction to classical dynamic logic, and even less to modal logic; it is a message to readers initiated in the art.

In this chapter we shall briefly outline the history and nature of dynamic modal logic, the directions and motives for its development, some unsolved and other currently unsolvable problems and an approach solving them.

1. *A Short History of PDL*

Propositional dynamic logic, PDL, in its currently most popular version was introduced by Fischer and Ladner (1979). They first defined PDL in 1977, as a propositional fragment of Pratt's (1976) Dynamic Logic. Pratt, on his part, invented DL in 1975, following R. Moore's suggestion to extend the Kripke modal approach to the before–after behaviour of computer programs. Moore's fruitful idea, on the other hand, appears to be a rediscovery (though cosmetically modified) of Salwicki's (1970) Algorithmic Logic. Salwicki's notion actually emerged in 1968, again in the USA, at Stanford (cf. (Salwicki, 1987)). Its author, taking no benefit from a possible worlds background, was stimulated by a paper of Engeler. Besides Engeler's, papers of Floyd, Hoare, and Naur influenced a number of related studies later on. A relevant pioneering work (cf. (Rasiowa, 1983)) is the forgotten book of Thiele (1966). The roots of the issue may also be traced back to the historical works of Turing and von Neumann. And even earlier: among the first known to investigate what is today known as modality were Democritus and Aristotle, 25 centuries ago.

2. *A Short Description of PDL*

Dynamic logics are formal systems intended for reasoning about actions in general. More precisely, they focus on action from a before–after (or, as process variants do, from a before–during–after) point of view, action and program being considered as synonyms in this context. Syntactically, in PDL, we have some atomic properties (to be evaluated as true or not of the states of some presumed universe) and some atomic programs (intended to supply some primary connections between these states). Starting from the atomic ones, formulae and programs of the PDL language are composed inductively, loosely speaking, via Boolean, modal, and regular rules.

For the semantics of PDL (and this is the gist of Moore's idea) Kripke models are chosen, associating to each program a separate modality. The universe having been fixed, the atomic formulae and programs are given arbitrary interpretation which is extended over the composed ones respecting the construction rules.

The deductive part of this formal system originates from Segerberg (1977). The soundness of Segerberg's system is obvious, while its completeness is not. Raised in 1977 by Segerberg, Gabbay, and Parikh, the question of completeness was of special interest until the brilliant proof of Kozen and Parikh (1981) appeared.

3. *Directions of PDL Studies and Motivation*

Since the axiomatics of Segerberg is recursively enumerable (r.e.), and the completeness obtained is in fact finite completeness (hence the non-theorems form an r.e. set as well), we have it that PDL is decidable. Its decision complexity is well known (viz. complete in deterministic exponential time), and this, together with the consequence problem, compactness, and interpolation—in rough outlines—exhausts the proper studies on the original version of PDL itself.

Mutations (extensions and variations) of PDL form another—apparently inexhaustible—branch of these investigations, with the main questions again being axiomatization/completeness and decidability/complexity, and, of course, comparison (as to expressiveness, descriptive power etc.) between these systems.

Consequently, several branches of autonomous studies were inspired: process logics and μ-calculus over PDL, dynamic algebras and temporal logics, probabilistic programs and logic of effective definitions, the non-standard approach of the Budapest group (see (Andreka *et al.* 1982)), the booming area of knowledge representation (see (Halpern, 1986)). The algorithmic prototype of DL, promoted by the Warsaw school, has been developing constantly since 1968 and is currently crowned with (Mirkowska and Salwicki, 1987).

The current state of the art has regularly been reviewed and reappraised: Parikh (1981), Goldblatt (1982), Parikh (1983), Harel (1984), Habasinski (1985), Goldblatt (1987), and Kozen and Tiuryn (1987) testify to steady progress, without collapses or revolutions.

Even a brief glance at the literature leaves the impression of a wave of intensive DL studies in the last two decades. What were the reasons? One folklore belief charges DLs with useful applications to computer practice. (We ourselves, not having touched such an application with our hands, do not plead such a cause.) Other views have it that the area provides a general mathematical theory about action, and a new insigth into the nature of computation. These views (of Harel (1984)) sound acceptable, the more so with (his) amendment to the latter, as to the *probable*, *indirect*, and *long-term* nature of *any* application.

Formally speaking, Dynamic Logic is *par excellence* logic, in particular, modal logic, and more precisely a poly-modal logic of mathematically organized modalities. So, Dynamic Logic is a natural superstructure of a natural generalization of a classical philosophical theory—the theory of modality. And this is a really good reason to chop logic of this sort.

4. *PDL—Hereditary Defective*

In formal systems, such as PDL, erected on the three pillars of abstractness: syntax, semantics, and axiomatics, the first and foremost question is that of completeness (soundness included). This question will be the leitmotiv in our paper too, and a mathematical background to our philosophy, which—being conceived and born in the bosom of tri-modal logic—will be explained on modal grounds too.

Having three modalities, $\langle R\rangle$, $\langle S\rangle$, and $\langle T\rangle$, it is well known what axiom to take guaranteeing that $T=R\cup S$, i.e., that the interpretation of the third is the union of the interpretations of the first two. This is the union-axiom scheme $\langle T\rangle A\leftrightarrow\langle R\rangle A\vee\langle S\rangle A$, for an arbitrary formula A, which "guarantees" the union in some very strong sense: it axiomatizes the union and, moreover, only the union, and on the other hand it modally defines the union. To recall the definitions, a set of formulae Γ *modally defines* some semantical class $\mathscr{E}$, if for each Kripke frame $\mathscr{F}$, $\mathscr{F}\in\mathscr{E}$ iff $\mathscr{F}\models\Gamma$; and Γ *axiomatizes* $\mathscr{E}$, if for each formula A, $\mathscr{E}\models A$ iff $\Gamma\vdash A$. The dynamic notation for the union-scheme is $\langle\alpha\cup\beta\rangle A\leftrightarrow\langle\alpha\rangle A\vee\langle\beta\rangle A$.

The intersection, in an irritating contrast with the union, does not have such good behaviour. A theorem of Goldblatt and Thomason (1975, Theorem 8)—and more precisely its folklore, cf. Gargov (1984–1986), polymodal form—says that the intersection $\cap$ is not modally definable, i.e., there is no set Γ of modal formulae such that for each Kripke frame $\mathscr{F} = \langle W, R, S, T \rangle$, $\mathscr{F} \models \Gamma$ iff $R \cap S = T$.

Concerning axiomatizability of $\cap$, there is a more positive result (we know such from Vakarelov and Tinchev, cf. (Vakarelov 1985–1988)), stating the completeness of the class of "intersectioned" models (i.e., those with $R \cap S = T$) with respect to the tri-modal logic, extended with the scheme $\langle T \rangle A \rightarrow \langle R \rangle A \wedge \langle S \rangle A$. However, this logic turns out to be also complete with respect to another, larger semantical class in which $T \subseteq R \cap S$. Thus, having a class which properly extends the class of "intersectioned" models, and which validates exactly the same modal formulae, there is no hope to modally axiomatize *exactly* the intersection.

These arguments also apply to PDL, where the intersection ($\alpha \cap \beta$—of the programs α, β) is of special interest, since it formalizes to a certain extent what is known as parallelism, or concurrency. Several recent studies (viz. Harel (1983, 1984), Parikh (1983), Danecki (1985), Peleg (1987)) take a special concern over the intersection, but—as could be expected in the light of the above arguments—neither proposes a syntactical characterization of intersection within the original frame of PDL.

Next at issue, after union and intersection, naturally comes the complementation of a modality, respectively of a program, in the spirit of Pratt's (1979) query for axiomatization and studying of "complemented dynamic algebras." Such a study would be, however, discouraged by a result of Harel (1984, Theorem 2.34) stating the undecidability of complemented PDL. Taking some goals here from our knowledge on intersection and union, namely that the union is an "easy" operation, and the intersection-"difficult," and in virtue of de Morgan's law $\alpha \cap \beta = \neg(\neg \alpha \cup \neg \beta)$, we expect the complement to be at least as difficult as the intersection is. Indeed, as shown by Gargov (Gargov 1984–1986), if we take—over bi-modal logic—the familiar S5 axioms for the union of the two modalities, we shall simultaneously axiomatize at least three different, and progressively included, classes of bi-modal frames $\langle W, R, S \rangle$:

(1) the class where $R \cup S$ is an equivalence relation,
(2) the class in which $R \cup S = W^2$, and
(3) the "interesting" complementary class, namely, where $R = \neg S$.

Moreover, Humberstone (1983) proposes an alternative axiomatics for (3), and an intermediate stage of his proof suggests that this class and

(4) the class of frames, where $R \cup S$ is an equivalence relation and $R \cap S = \varnothing$,

validate the same modal formulae. So, as shown by the arguments of Humberstone and Gargov, the axiomatization of the complementary class over the bi-modal language is a hopeless case: this modal class is axiomatically inseparable from the other three mentioned above.

Particularly, for modal definability, we can quote an elegant construction of Humberstone (1983) which makes no explicit reference to the heavy artillery of Goldblatt and Thomason's theorem, and shows the complementary class as modally undefinable, over the usual bi-modal language.

Following this line, after $\cup$, $\cap$, $\neg$, we shall surely arrive at inclusion $\subseteq$ and equality $=$ of modalities, respectively of programs. (We say that $x \models R \subseteq S$ if $\forall y(xRy \rightarrow xSy)$.) In the bi-modal case, we have again a result of Tinchev and Vakarelov (Vakarelov, 1985–1988) showing the completeness of the system containing the axiom scheme $(R \subseteq S) \rightarrow (\langle R \rangle A \rightarrow \langle S \rangle A)$ and a rule, which can be loosely formulated as "From $\langle R \rangle p \rightarrow \langle S \rangle p$ infer $R \subseteq S$."

The theoretical mainstream which naturally covers the above operations ($\cup$, $\cap$, $\neg$) and predicates ($\subseteq$, $\not\subset$, $=$, $\neq$) as special cases, drives at a syntactical description of first-order definable operations, manageable in dynamic modal logic. Another branch within the same stream would eventually aim to fill an old gap in modal logic, between quantificational and propositional studies. This question is not only technical, but rather methodological: how to introduce the letter of the quantifier while keeping the initial spirit of the modality? No commonly accepted answer to that question has been proposed yet.

Further on, the next theorist's dream will be to increase the order, arriving at second-order definable operations, which seem to be quite natural here, in the context of the historical relations between modal/dynamic logic and second-order logic. At this stage we enter another branch of PDL studies, namely those of "recursive," or "repetitive," or "μ-" character. A representative puzzle in that area—modal description of second-order definable operations—is Streett's (1982) conjecture, still unproved in its original setting, for axiomatization of the "repeating" predicate (which is true of a state x of $\langle W, R \rangle$, if an infinite R-path exists in x). Thus, we gradually arrived at the limits of a natural problem, even the most natural special cases of which manifest themselves as hard nuts to crack.

5. *Removing the Defect: Formal Speculations*

Let us examine more carefully the intersection-modality $[R \cap S]$. The semantical condition is $x \models [R \cap S]p$ iff $\forall y(xRy \;\&\; xSy \rightarrow y \models p)$. Having the conjuncts xRy and xSy in the antecedent, the quantifier $\forall y$ does not distribute over them, and thus the semantical condition for $[R \cap S]$ is not reducible to the conditions for $[R]$ and $[S]$. So, in search for modal description of the intersection, one possible solution would be to reverse Kripke's semantical condition, defining a new modality $[\![R]\!]$, or ◫, with $x \models [\![R]\!]\, p$ iff $\forall y(y \models p \rightarrow xRy)$. Philosophically, the "box" $\Box$ represents *necessity*, to exactly the same extent to which the "window" operator ◫ fits the metamathematics of *sufficiency*. We would now obtain for the intersection $R \cap S$ a condition with conjuncts in consequent, which the quantifier distributes over

$$\begin{aligned} x \models [\![R \cap S]\!]\, p \quad & \text{iff} \quad \forall y(y \models p \rightarrow xRy \;\&\; xSy) \\ & \text{iff} \quad \forall y(y \models p \rightarrow xRy) \;\&\; \forall y(y \models p \rightarrow xSy) \\ & \text{iff} \quad x \models [\![R]\!]\, p \wedge [\![S]\!]\, p, \end{aligned}$$

whence the formula $[\![R \cap S]\!]\, p \leftrightarrow [\![R]\!]\, p \wedge [\![S]\!]\, p$, which turns out to represent the intersection pretty well, at least as well as the union-axiom represents the union; see (Gargov *et al.*, 1987). Of course, for union's sake the original modality $[R]$ has to be kept in the vocabulary as well. Not only union and intersection become equipollent under such scope, but a successful treatment of the complementation of modalities is already possible, as a side effect:

$$[\neg R]A \leftrightarrow [\![R]\!] \neg A \qquad \text{and} \qquad [\![\neg R]\!]A \leftrightarrow [R] \neg A.$$

The traces of such an approach—associating with each relation R two modalities $[R]$ and $[\![R]\!]$—lead back to van Benthem (1979), Humberstone (1983), and Tehlikeli (1985), and hints in that sense are given by Vakarelov and by Goldblatt as early as 1974. The idea was later launched and largely discussed in (Gargov *et al.*, 1987), (Humberstone, 1987a), (Goranko, 1987, 1989), and (Gargov and Passy, 1989).

We saw above the "alternative" modality ◫ growing out of the intention to reduce the modality-intersection to modalities of the intersects. Let us try the other way round, to restore the intersection $[R \cap S]$, provided $[R]$ and $[S]$ were given:

$$\begin{aligned} x \models \langle R \rangle p \wedge \langle S \rangle p \quad & \text{iff} \quad \exists y(xRy \;\&\; y \models p) \;\&\; \exists z(xSz \;\&\; z \models p) \\ & \text{iff} \quad \exists y\, \exists z(xRy \;\&\; xSz \;\&\; y \models p \;\&\; z \models p). \end{aligned}$$

If we had modal means guaranteeing the identity of states, viz. $y = z$, we would have been able to ensure $x \models \langle R \cap S \rangle p$. Alas, "we have no means of explicitly describing the possible worlds" (Fagin and Vardi, 1985). But we may coin such means exploiting more of $y \models p \;\&\; z \models p$. Restricting some of the propositional variables p (say, only the even-indexed in the alphabet $\Phi = \{p_1, p_2, \ldots\}$) to be interpreted as true at exactly one state (i.e., as proper names for these states), we already have $y \models p_2$ & $z \models p_2$ imply $y = z$.

For clarity's sake, we pick a new alphabet $\Sigma = \{c_1, c_2, ...\}$ (instead of $\{p_{2k}/k < \omega\}$) and require that the c_k's be interpreted as proper names for the states of the universe, i.e., that

(i) each name nominate exactly one state

(ii) each state have at least one name.

The goal has been reached. The scheme $\langle R \cap S \rangle c \leftrightarrow \langle R \rangle c \wedge \langle S \rangle c$, for $c \in \Sigma$, both modally defines (which is clear by inspection) and axiomatizes (which will be proven in the next chapter) the intersection of modalities. Similarly, one can treat the negation $\neg R$, via $\langle \neg R \rangle c \leftrightarrow [R] \neg c$. Of course, a proper axiomatic pad for the conditions (i) and (ii) is presumed beforehand, to which end we first of all add an extra S5 modality $[v]$ and interpret v as the Cartesian square of the universe. We call the modal systems thus equipped (with names and universe-modality) combinatory systems, from where the abbreviation CPDL comes. CPDL solves the problems.

Thus we state as a main defect of Kripke modal systems the existence of explicit *individual* worlds s, t, ... in the semantics vs. representatives of *collections* of worlds p, q, ... on the syntactical side where therefore the world's individuality is lost. Christening the worlds, the combinatory approach does remove this defect and its unpleasant consequences. A consequence of the sort, called in (Fagin and Vardi, 1985) "flabbiness" of Kripke models, is the existence of semantically (modally) equivalent Kripke models which are not isomorphic. No flabby combinatory models are available, rejected by a version of Scott's $L_{\omega_1\omega}$-isomorphism theorem, Section I.2.

Fagin and Vardi's paper presents a good collection of critical remarks on the state of affairs stated after Kripke, and the authors suggest weakening the semantics until it fits the syntax, as opposed to the policy we outlined above—enriching the syntax to fit the semantics. Another defender of the alternative policy is Peleg (1987), who proposes a Montague type semantics for dynamic logic, focusing the semantics on collections of states, rather than on individuals. By happenstance, Peleg seems to have also been provoked by the famous intersection of programs.

6. *Yet in the Philosophy: The Quasi-Henkin Completeness Proof*

The completeness proof for PDL is a hereditary acquisition from modal logic, and the modal completeness proof, in its most popular versions, is an imitation of Henkin's completeness proof for the predicate calculus. This last proof is essentially based on the notion of (maximal) consistent set of formulae, or the synonym—(maximal) theory. The so called Henkin completeness proof flourishes on most of the popular modal systems. To describe it, Humberstone (1983)—axiomatizing what we called the $\Box$-$[\![\ ]\!]$ system—writes: "The argument we use is modeled after Cresswell's adaptation to modal logic of the method Henkin used to prove completeness of first-order logic, rather than the more widely known adaptation of that method due to Scott and Makinson... In the former case maximal theories are correlated with elements of the falsifying model but the correlation is not required to be one–one, so that there is more freedom in constructing the required accessibility relation than on the latter approach—as generally implemented—in which the maximal theories are identified with the points of the model serving to falsify any given nontheorem."

These two modal adaptations of the Henkin proof share the same deviation from their prototype. This last, cf. (Shoenfield, 1967), zooms the negation of the disprovable formula up to one maximal theory. The points of the counter-model are then defined, modulo this fixed theory, as equivalence classes of constant (variable-free) terms, with no reference to another maximal theory. In contrast, the modal proofs appeal to much more than one, sometimes to all maximal theories, which themselves have to serve for points of the counter-model: the modal syntax, as it stands, proves too poor to offer a counterpart of constant terms, which are the pillars of the Henkin model's worlds. (Of course, a default understanding nominates a theory as the infinite conjunction of all its elements, but such unfinitary conjunctions are not in the syntactical arsenal.)

Thus, the main deviation of the modal completeness proof from Henkin's comes to overcome the misfit between the incapability of the modal syntax and the capacity of its Kripke semantics. The same concern has been considerably aggravated as soon as the mutations of PDL are considered, and the proof requires non-trivial intellectual efforts in some of these cases (e.g., for Deterministic PDL, see (Ben-Ari *et al.*, 1982)). So the syntactico-semantical misfit makes the completeness proof not automatically applicable to mutations of modal logic, which appears to be a handicap not of the proof but of the logic.

The enriched $\Box$-$[\![\ ]\!]$ modal language more or less solves the syntactical troubles caused by the intersection and complement. (The completeness proof for $\Box$-$[\![\ ]\!]$, modulo some extra complications, resembles that for S5.) However, this attempt proves not satisfactory enough to compensate for the "big misfit": the $\Box$-$[\![\ ]\!]$ models are still flabby. This is not surprising, since $[\![R]\!]$ is a synonym of $[\neg R]$, and this syntactical means is not devised for handling a particular possible world.

The other solution, to enrich the modal language with proper names for the possible worlds, is superior to the former. The names solve the syntactical problems around the Boolean operations on modalities/programs, and they do this as a matter of course. The presence of names, moreover, permits restoration of the original Henkin proof, the names starring as special Henkin constants (see Section 1.5). One takes only one maximal theory, and the points of the falsifying model come out as equivalence classes of names, modulo this theory. Implicitly, each point does correspond to a different maximal theory. However, each theory contains at least one name—this is guaranteed by the deductive system—and this name is as distinctive as a fingerprint for this theory. Thus the name serves in this case as the lacking infinitary conjunction in the language.

This restored Henkin proof has, not surprisingly, an additional advantage: it works equally smoothly for all of the popular modal/dynamic mutations. Moreover, it resists tempting mathematical generalizations: completeness theorem for all expressible extensions (Section II.2.1), completeness theorem for all existential ($\exists^*$-) and universal ($\forall^*$-) definitional extensions of CPDL (Section II.2.2), and completeness theorem for all first-order, i.e., $\{\exists, \forall\}^*$-definitional extensions of the Quantified CPDL (Section III.2).

Here, forestalling the events, we arrived to quantification, which is another mathematical attraction the names provide. We simply let the quantifiers range over the names and obtain one more than natural modal quantificational theory, developed in Chapter III.

In modal logic, besides axiomatizability, there are a number of notions and questions of interest: decidability, modal equivalence (local and global), modal definability (of semantical classes and of classical formulae), first- and higher-order definability (of modal formulae), expressibility. Most of these questions are multiplied—when considered in states, models, and frames, finite or not—and the interrelations between them create a real terminological jungle, into which the combinatory approach claims to put some clearer order.

7. *The Present Paper*

The present paper presents Combinatory PDL as a fusion of two ideas featured in our Ph.D. Theses: names in modal logic (Passy, 1984), and ω-axiomatics in dynamic logic (Tinchev, 1986).

The first draft, completed by the end of 1984, was designed as the "full paper" promised in our preliminary reports on the matter (Passy and Tinchev, 1984, 1985a, b]. We were than planning an opusculum containing detailed proofs and stimulating some future investigations on the subject. However, due to different circumstances, we prepared the final revision required by "Information and Computation" (then still "Information and Control") as late as the fall of 1989. Meanwhile some of these investigations appeared: [Gargov, 1986; Gargov and Passy, 1985, 1988; Petkov, 1987; Radev, 1985, 1986, 1987; Sotirov, 1984, 1985; Tehlikeli, 1985; Tinchev, 1988], and to complete this list of papers we sentimentally love, we should add [Gargov *et al.* 1987, 1989; Goranko, 1987; Tinchev and Vakarelov, 1985]. We were thus enlightened on the subject and it is natural that this enlightenment should be reflected here. That is how the present paper, originally planned as an opusculum, grew (along with us) into a magnum opus.

Though the paper can be regarded as a compendium on combinatory dynamic logics, much of the material is published here for the first time. Such is, in particular, the axiomatization-through-expressiveness idea, which became the backbone of the axiomatizability results, and hence of the whole paper.

8. *Notes on Terminology and Notation*

We keep independent enumeration of heads (definitions, theorems, notes, etc.) in different sections, and quote, as usual, omitting chapter number when quoting within the same chapter. The equality mark $=$ also stands for graphical identity of two words; $c \,\mathcal{J}\, A$ ($c \,\not\mathcal{J}\, A$) denotes that the letter c occurs (does not occur) in the word A. The composition marks $\circ$ (for relations) and ; (for programs) will be omitted as usual, the composition $\alpha; ...; \alpha$ of $k \geqslant 0$ times α will be denoted by α^k, where α^0, denoted by ι, is the identity, or diagonal relation on the universe. The power-set of a set M will be denoted by $\mathscr{P}(M)$; ω is the cardinality of the set of natural numbers N; $M_{/\sim}$ is the quotient of the set M modulo the equivalence relation $\sim$. For a binary relation R, $R(s) =_{\mathrm{DF}} \{t/sRt\}$. By PC we shall mean the predicate calculus, in a version specified in convention I.1.5 below.

Fixing the logical lexicon $\neg$, $\vee$, $\exists$, $\langle\ \rangle$, the standard abbreviations will be in force: $\wedge$, $\rightarrow$, $\leftrightarrow$, 0, 1, $\forall$, $[\ ] = \neg\langle\ \rangle\neg$; $\odot$ and $\boxdot$ will stand for $\langle v\rangle$ and $[v]$.

Throughout the paper, having fixed some modal system $\mathscr{E}$, $\mathscr{E} \vdash$, or $\vdash_{\mathscr{E}}$, or even $\vdash$, will stand for provability in $\mathscr{E}$, whereas $\mathscr{M}$ and $\mathscr{N}$ will be typical letters for models of $\mathscr{E}$. The common semantical dictionary will be used: for a model $\mathscr{M}$ with universe M, state $s \in M$, valuation V and for a modal formula A, by $\mathscr{M}, s \models A$ we denote that s *satisfies* A, i.e., that $s \in V(A)$. We say that $\mathscr{M}$ *is a model* for A, denoted $\mathscr{M} \models A$, if $\forall s_{s \in M}(\mathscr{M}, s \models A)$; $\mathscr{M}$ *is a model* for the set of formulae Γ, denoted $\mathscr{M} \models \Gamma$, if $\forall A_{A \in \Gamma}(\mathscr{M} \models A)$; A *is satisfiable* if $\exists \mathscr{M}, s\ (\mathscr{M}, s \models A)$; A *is valid*, denoted $\models A$, if $\forall \mathscr{M}(\mathscr{M} \models A)$. The models $\mathscr{M}$ and $\mathscr{N}$ are said to be *modally equivalent* (over the modal language $\mathscr{L}$), denoted $\mathscr{M} \models\!\dashv \mathscr{N}$, if $\forall A_{A \in \mathscr{L}}(\mathscr{M} \models A$ iff $\mathscr{N} \models A)$. And we say that $\mathscr{N}$ are *isomorphic*, denoted $\mathscr{M} = \mathscr{N}$ if they are copies of the same model; the isomorphic models will be always identified, justifying the notation $\mathscr{M} = \mathscr{N}$.

The fmp is abbreviation for "finite model property," and ih for "inductive hypothesis."

Two more notions, cf. (Shoenfield, 1967): A *simple extension* of a deductive set of formulae is a new, bigger set over the same language, whereas a *definitional extension* assumes extension of the ground language.

Chapter I: Combinatory PDL

In this chapter we give a fairly detailed presentation of what we call the Combinatory approach to dynamic modal logic; throughout, special attention will be payed to CPDL$^\cap$—a sample extension (with intersection) of the basic system CPDL. The results on CPDL (Sections 1–6) are easily extendable over CPDL$^\cap$, exclusion being made by the decidability result (Section 7). This care of CPDL$^\cap$ serves as a prologue to the next two chapters, written more sketchily, where the merits of the approach will be made use of, to handle numerous mutations of the basic system.

1. *Language, Semantics, Expressiveness*

Let Σ, Φ_0, and Π_0 be three countably infinite and pairwise disjoint alphabets, known respectively as the set of *names* (or *constants*), the set of *atomic propositions*, and the set of *atomic programs*. The letter $v \notin \Sigma \cup \Phi_0 \cup \Pi_0$ will be called the *universe* program (or modality).

DEFINITION 1.1. The *language* of CPDL consists of *formulae* and *programs*, inductively defined by

(i) The elements of $\Sigma \cup \Phi_0$ are formulae. The elements of $\Pi_0 \cup \{v\}$ are programs.

(ii) If A, B are formulae and α, β are programs, then:

$\neg A$, $A \vee B$, $\langle \alpha \rangle A$ are formulae, and $\alpha; \beta$, $\alpha \cup \beta$, α^*, $A?$ are programs.

For CPDL$^\cap$ we add the clause

"$\alpha \cap \beta$ is a program."

The set of all CPDL formulae is denoted by Φ, and the set of all programs by Π. Typical elements are c, d, e—for Σ; a, b—for Π_0; α, β, γ—for Π; p—for Φ_0; A, B, C—for Φ; and Γ—for $\mathscr{P}(\Phi)$.

Abbreviations. $\Diamond A =_{DF} \langle v \rangle A$, $\Box A =_{DF} [v]A$, $\iota =_{DF} 1?$, $\hat{c} =_{DF} v; c?$, $\langle c \rangle =_{DF} \langle \hat{c} \rangle = \langle v; c? \rangle$, $[c] =_{DF} [\hat{c}] = [v; c?]$.

DEFINITION 1.2. A *model* for CPDL is a quadruple $\mathcal{M} = (M, R, \chi, V)$, where M is a non-empty set (universe of states or worlds), and the other three are functions

$$R: \Pi \to \mathscr{P}(M^2),\ \chi: \Sigma \xrightarrow{\text{onto}} M, \text{ and } V: \Phi \to \mathscr{P}(M)$$

connected by (where $s \models A$ is $s \in V(A)$, and $sR_\alpha t$ is $(s, t) \in R(\alpha)$)

$$V(c) = \{\chi(c)\}, \qquad \text{for} \quad c \in \Sigma,$$

$$V(\neg A) = M \setminus V(A), \qquad V(A \vee B) = V(A) \cup V(B),$$

$$V(\langle \alpha \rangle A) = \{s / \exists t (sR_\alpha t \ \&\ t \models A)\},$$

$$R_v = M^2,$$

$$R_{\alpha \cup \beta} = R_\alpha \cup R_\beta,$$

$$R_{\alpha\beta} = R_\alpha R_\beta = \{(s, t) / \exists v (sR_\alpha v \ \&\ vR_\beta t)\},$$

$$R_{\alpha^*} = (R_\alpha)^* = \bigcup_{k < \omega} (R_\alpha)^k,$$

$$R_{A?} = \{(s, s) / s \models A\}.$$

For CPDL$^\cap$ we add $R_{\alpha \cap \beta} = R_\alpha \cap R_\beta$.

EXERCISES 1.3. 1. $\mathcal{M}, s \models \Box A$ iff $\mathcal{M} \models A$

2. $\mathcal{M}, s \models \langle c \rangle A$ iff $\mathcal{M} \models [c] A$ iff $\mathcal{M}, \chi(c) \models A$
3. $\mathcal{M} \models \langle c \rangle d$ iff $\chi(c) = \chi(d)$
4. $\models \langle c \rangle A \leftrightarrow [c] A$.

Note 1.4. The restrictions of CPDL's models to PDL's language (i.e., to the language without v and Σ) yield precisely the *countable* PDL models. The interpretations $\chi(c)$ and $R(v)$ justify the term "name" for c and "universe" for v. The term "constant" for c comes after the semantics of $\hat{c}$: $R(\hat{c}) = M \times \{\chi(c)\}$, which is a totally defined constant function. These constants provide an equal alternative for CPDL's syntax: the set $\Sigma^\wedge = \{\langle c \rangle / c \in \Sigma\}$ of modalities may successfully replace the set Σ of propositions. Skordev (1980) uses similar constant functions in some recursive-theoretic context, and we adopted his Combinatory space as godfather of our modal system.

The surjectivity of χ imposes one prima facie drastic limitation on the models: they are at the most countable. However, as demonstrated in Sections 6 and 7 below, this is a restriction only apparently and is therefore an excellent price for the *strong separability* of the models thus achieved. Given a model $\mathcal{M} = (M, R, \chi, V)$, for each state $s \in M$, there is some formula, namely any $c \in \chi^{-1}(s)$, separating s from *all* other worlds of that model:

$$\forall t_{t \in M}(t \models c \text{ iff } t = s), \qquad \text{i.e.,} \quad V(\neg c) = \{t/t \neq s\}.$$

We call this separability "strong" since it is stronger than the usual one,

$$\forall s\, \forall t_{t \neq s}\, \exists A(s \models A \;\&\; t \models \neg A),$$

and it is this strong separability that enables easy syntactic treatment of world's identity (see Exercise 1.3(3)), which in its turn will be the essence of what follows in this paper.

The semantics of v agrees quite well with the definitions of satisfiability and validity in a model, and so the presence of v increases the metatheoretical capacities of the basic language. We have

(a) A is satisfiable iff $\Diamond A$ has a model; and A has a model iff $\boxdot A$ is satisfiable (so the problems of "having a model" and "satisfiability" are co-reducible over CPDL).

(b) In the terminology of (Parikh, 1981; Harel, 1984) the *global consequence* problem, $A \models_g B$ if $\forall \mathcal{M}$ ($\mathcal{M} \models A$ only if $\mathcal{M} \models B$), becomes reducible to (and hence co-reducible with) the *consequence* problem, $\models A \rightarrow B$: $A \models_g B$ iff $\models \boxdot A \rightarrow \boxdot B$.

(c) In particular, one can smoothly treat the partial correctness assertion PCA—$A\{\alpha\}B$, cf. (Parikh, 1981)—without extra semantical means: $A\{\alpha\}B$ has the semantics of $\boxdot(A \rightarrow [\alpha]B)$. (Conversely, $\boxdot$ could be contextually defined in PCA's language as well: $\boxdot B =_{\mathrm{DF}} 1\{1?\}B$).

In the next section we shall pay special attention to the relationship between the combinatory models and language. In what remains of this section we given some definitions frequently used in the sequel, and establish some results on the expressive power of the CPDL language, relative to the language of the predicate calculus, PC.

Notational Convention 1.5. By PC we denote the predicate calculus with equality ($=$), with no function symbols, and with binary and unary predicates only. Let F, G be typical letters for PC formulae; we assume PC formulae in prenex form $Q_1 e_1 \cdots Q_k e_k G$, where $Q_j \in \{\forall, \exists\}$, and G is *open* (quantifier-free). The letters α and p will range over binary and unary predicates, respectively, and c, d, e over the individual variables; the modal and first-order use of these letters will be easily distinguishable. We shall sometimes abbreviate $Q_1e_1 \cdots Q_ke_kG(\alpha, ..., p, ..., e_1, ..., e_k, c, ...)$ as $\mathbf{Qe}\, G(\boldsymbol{\alpha}, \mathbf{p}, \mathbf{e}, \mathbf{c})$.

By PC $\vdash F$ we denote that F is a theorem of PC. By $\mathcal{M} \Vdash F$ we denote that the PC formula $F = F(\alpha, ..., p, ..., c, ...)$ holds in the model $\mathcal{M} = (W, R, \chi, V)$, i.e., that the assertion made by F for the relations $R_\alpha, ...,$ the predicates $V(p), ...$ and the states $\chi(c), ...,$ holds in $\mathcal{M}$. So, in $\mathcal{M}$, we interpret the PC predicate α as the CPDL program α, p as the CPDL proposition p, and the (free) variable c—as the constant $\chi(c)$. So, the PC model $\mathcal{M}$ assumes also the valuation χ of the variables. As usual, $\Vdash F$ stands for $\forall \mathcal{M}\ (\mathcal{M} \Vdash F)$.

Note 1.6. The restrictions we pose on PC are only cosmetical and do not spoil the results we shall be referring to: completeness theorem, PC $\vdash F$ iff $\Vdash F$, and Σ_1^0-completeness for deciding provability in PC.

DEFINITION 1.7. Let F have k free variables. Let A have k names appearing in it. We say that A *expresses* F, if, for each model $\mathcal{M}$, and each $c_1, ..., c_k$, $\mathcal{M} \Vdash F(c_1, ..., c_k)$ iff $\mathcal{M} \models A(c_1, ..., c_k)$. Since A expresses F iff $\boxdot A$ does, we shall always suppose that the expressive formula begins with $\boxdot$.

Note that if A expresses F, then $\Vdash F$ iff $\models A$. Obviously, the class of expressible PC formulae is closed under conjunction, negation, and disjunction.

We shall show in the next chapter that if A expresses F, then A axiomatizes the operation (on programs appearing in F) given by the k-place predicate

$$R_F = \{(\chi(c_1), ..., \chi(c_k)) \,/\, \mathcal{M} \Vdash F(c_1, ..., c_k)\}.$$

DEFINITION 1.8. Let $\tilde{}$ be the translation from open PC formulae into CPDL formulae, $G \mapsto G^\sim$, which replaces

(a) each occurrence of $\alpha(c, d)$ by $\langle c \rangle \langle \alpha \rangle d$, for each $\alpha \neq =$;

(b) each occurrence of $c = d$ by $\langle c \rangle\, d$;

(c) each of occurrence of $p(c)$ by $\langle c \rangle\, p$.

EXPRESSIVENESS THEOREM 1.9. *For each open PC formula G, $G^\sim$ expresses G.*

Proof. Straightforward induction on the construction of G, cf. (Tinchev, 1986). ∎

EXAMPLES 1.10. Open formulae, though a narrow class, suffice for defining some familiar operations on programs, which therefore are expressible over CPDL:

union: $\alpha(c, d) \vee \beta(c, d)$

expressed by $\textcircled{c}\langle\alpha\rangle d \vee \textcircled{c}\langle\beta\rangle d$

intersection: $\alpha(c, d) \wedge \beta(c, d)$

expressed by $\textcircled{c}\langle\alpha\rangle d \wedge \textcircled{c}\langle\beta\rangle d$

complementation: $\neg\alpha(c, d)$

expressed by $\neg\textcircled{c}\langle\alpha\rangle d$

converse: $\alpha^{-1}(c, d)$

expressed by $\textcircled{d}\langle\alpha\rangle c$.

Theorem 1.9 gives only what is seen at first glance w.r.t. CPDL/PC expressiveness. For, e.g., the composition $\alpha;\ \beta(c, d)$ is also expressible (by $\textcircled{c}\langle\alpha\rangle\langle\beta\rangle d$), and it might therefore swallow one existential quantifier. We shall make use of this fact in the larger dynamic language including negation and intersection of programs.

EXPRESSIVENESS THEOREM 1.11. *Each PC formula which has no nested quantifiers is expressible in the language of* $CPDL + \{\neg, \cap\}$.

Proof. Since the expressible PC formulae are closed under conjunction, disjunction, and negation, it will be representative enough to express a formula of the sort

$$G(c_1, c_2, c_3, c_4) = \exists c(R_1(c_1, c) \wedge \neg R_2(c_2, c) \wedge R_3(c, c_3) \wedge \neg R_4(c, c_4) \wedge R_5(c, c) \wedge \neg R_6(c, c) \wedge P(c)),$$

which is expressed by

$$G^{\sim}(c_1, c_2, c_3, c_4) = \langle(v; c_1?; \alpha_1 \cap v; c_2?; \neg\alpha_2); (\iota \cap R_5 \cap \neg R_6 \cap P?); (\alpha_3; c_3?; v \cap \neg\alpha_4; c_4?; v)\rangle 1. \quad \blacksquare$$

Unfortunately, even in the presence of the other familiar algebraic-relational operations in the dynamic language this result cannot be spread over PC formulae with even two nested existential quantifiers. We have:

PROPOSITION 1.12. *Let* R_j *be* $R(\alpha_j)$ *for* $j = 1, ..., 6$. *Let* α *be a program with interpretation* R_α *given by*

$$xR_\alpha y \leftrightarrow \exists z\, \exists t(xR_1 z\ \&\ zR_2 y\ \&\ xR_3 y\ \&\ xR_4 t\ \&\ tR_5 y\ \&\ zR_6 t).$$

Then there is no program β in the dynamic language of $;$, $\cup$, $\cap$, $\neg$, ${}^{-1}$, $\subseteq$, $=$, $?$, $*$, ν, ι, Σ, $\alpha_1, \ldots, \alpha_6$ *such that* $R_\alpha = R_\beta$, *in each model.*

Proof. Cf. (Tinchev, 1986). ∎

In Chapter III, we shall obtain a result showing that each PC formula is expressible in the Quantified CPDL, CDL. It is folklore nowadays that the iteration * is not expressible in PC, and so to establish more accomplished expressiveness results we should either give up the * on the modal side, or add some transitive closure operator on the PC side, cf. [Tiomkin and Makowski, 1985), or see earlier references (pointed out by the referee) (Aho and Ullman, 1979; Zloof, 1976). We leave these questions on expressibility open restricting ourselves to one formulation:

Problem 1.13. Give an explicit description of the sets:

$\{F \in \mathrm{PC}/F$ is expressible in CPDL$\}$

$\{F \in \mathrm{PC}/F$ is expressible in CPDL $+ \{\cap, \iota, \neg, {}^{-1}, \subseteq\}\}$

$\{A \in \mathrm{CPDL}/A$ expresses some PC formula$\}$

$\{A \in \mathrm{CPDL} + \{\cap, \iota, \neg, {}^{-1}, \subseteq\}/A$ expresses some PC formula$\}$.

2. *Language vs. Models: Scott's Isomorphism Theorem*

As noted by Fagin and Vardi (1985), Kripke models for modal logic are "flabby" in the sense that two non-isomorphic Kripke models can be modally equivalent. Indeed, from general modal theory, it follows that each model is modally equivalent to any of its disjoint degrees (i.e., the disjoint union of a number of its isomorphic copies), which, as a rule, are not isomorphic to the original model. Moreover, there is another natural equivalence, the "local" modal equivalence of models, much stronger than the usual one, but still weaker than isomorphism; see (Parikh, 1981, Sect. 6). We show in this section that, thanks to the names, modal equivalence guarantees isomorphism in the Combinatory case, and that, thanks to ν, local and usual modal equivalence of models coincide.

DEFINITION 2.1. An *elementary formula*, EF, is any of the types

$$\text{Ⓒ}\, d,\ \text{Ⓒ}\, p,\ \text{Ⓒ}\, \langle a \rangle d, \text{ where } c, d \in \Sigma,\ a \in \Pi_0,\ p \in \Phi_0.$$

DEFINITION 2.2. (i) An *elementary set* is any set Γ of EF's such that for all c, d, $e \in \Sigma$, $p \in \Phi_0$, and $a \in \Pi_0$,

$\cdot$) $\quad$ Ⓒ $c \in \Gamma$, and

if $\Diamond\textcircled{c}\, d \in \Gamma$, then:

$\cdot$) $\Diamond\textcircled{d}\, c \in \Gamma$

$\cdot$) $\Diamond\textcircled{d}\, e \in \Gamma$ implies $\Diamond\textcircled{c}\, e \in \Gamma$

$\cdot$) $\Diamond\textcircled{c}\, p \in \Gamma$ implies $\Diamond\textcircled{d}\, p \in \Gamma$

$\cdot$) $\Diamond\textcircled{c}\, \langle a \rangle e \in \Gamma$ implies $\Diamond\textcircled{d}\, \langle a \rangle e \in \Gamma$

$\cdot$) $\Diamond\textcircled{e}\, \langle a \rangle c \in \Gamma$ implies $\Diamond\textcircled{e}\, \langle a \rangle d \in \Gamma$.

(ii) if Γ is elementary, then the set $\Gamma \cup \{\neg A / A$ is EF & $A \notin \Gamma\}$ is called a *full set* (*over* Γ).

DEFINITION 2.3. For a given model $\mathcal{M} = (M, R, \chi, V)$, the full set over the set

$$\{\Diamond\textcircled{c}\, d/\chi(c) = \chi(d)\} \cup \{\Diamond\textcircled{c}\, p/\chi(c) \in V(p)\} \cup \{\Diamond\textcircled{c}\, \langle a \rangle d/\chi(c)\, R_a \chi(d)\}$$

is called a *diagram of* $\mathcal{M}$, denoted diag($\mathcal{M}$).

Obviously, diag($\mathcal{M}$) is recursive in $\mathcal{M}$ (in the familiar meaning of this notion), and

(1) $\mathcal{M} \models \text{diag}(\mathcal{M})$.

DEFINITION 2.4. For a full set Γ, we define a model mod(Γ) = (M, R, χ, V) as follows. Let $c \sim d$ stand for $\Diamond\textcircled{c}\, d \in \Gamma$, which is obviously an equivalence reelation, and let $|c| =_{\text{DF}} \{d/c \sim d\}$. Then we set

$$M =_{\text{DF}} \Sigma_{/\sim}, \quad \chi(c) =_{\text{DF}} |c|, \quad V(p) =_{\text{DF}} \{|c|/\Diamond\textcircled{c}\, p \in \Gamma\},$$
$$R_a =_{\text{DF}} \{(|c|, |d|)/\Diamond\textcircled{c}\, \langle a \rangle d \in \Gamma\},$$

and define R_α and $V(A)$ as the usual inductive extensions respecting the definition of model.

The fact that mod(Γ) is indeed a model is justified by the definition of elementary sets, which implies that $\sim$ is an equivalence relation correct to R_a and $V(p)$.

It is clear by inspection that for an arbitrary full set Γ and a model $\mathcal{M}$,

(1*) $\text{mod}(\Gamma) \models \Gamma$

(2) $\mathcal{M} \models \Gamma$ implies $\Gamma = \text{diag}(\mathcal{M})$

(2*) $\mathcal{M} \models \Gamma$ implies $\text{mod}(\Gamma) = \mathcal{M}$;

hence, by (1, 1*, 2, 2*), we get

(*) $\mathcal{M} \models \Gamma$ iff $\Gamma = \text{diag}(\mathcal{M})$ iff $\text{mod}(\Gamma) = \mathcal{M}$.

From (2*), with $\Gamma = \text{diag}(\mathcal{M})$, and (1) we obtain

(3) $\text{mod}(\text{diag}(\mathcal{M})) = \mathcal{M}$.

From (2), with $\mathcal{M} = \text{mod}(\Gamma)$, and (1*) we obtain

(3*) $\text{diag}(\text{mod}(\Gamma)) = \Gamma$.

Thus we arrive at the following:

CORRESPONDENCE THEOREM 2.5. *There exists a bijective correspondence* diag $(=\text{mod}^{-1})$ *between models and full sets such that for each model* $\mathcal{M}$ *and full set* Γ,

$$(*)\ \mathcal{M} \models \Gamma \textit{ iff } \Gamma = \text{diag}(\mathcal{M}) \textit{ iff } \text{mod}(\Gamma) = \mathcal{M}.$$

SCOTT'S ISOMORPHISM THEOREM 2.6. $\mathcal{M} = \mathcal{N}$ *iff* $\mathcal{M} \models \text{diag}(\mathcal{N})$.

Proof. (if) Let $\mathcal{M} \models \text{diag}(\mathcal{N})$. Hence, by (2), $\text{diag}(\mathcal{N}) = \text{diag}(\mathcal{M})$ and therefore $\text{mod}(\text{diag}(\mathcal{N})) = \text{mod}(\text{diag}(\mathcal{M}))$. So, by (3), $\mathcal{N} = \mathcal{M}$.

(only if) $\mathcal{M} = \mathcal{N}$ guarantees $\text{diag}(\mathcal{M}) = \text{diag}(\mathcal{N})$, and by (1), $\mathcal{M} \models \text{diag}(\mathcal{N})$. ∎

COROLLARY 2.7. $\mathcal{M} \models\!\dashv \mathcal{N}$ *only if* $\mathcal{M} = \mathcal{N}$.

The similarity between the above theorem and Scott's (1963) $L_{\omega_1\omega}$ isomorphism theorem was pointed out to us by Slavjan Radev. The difference is that Scott's $L_{\omega_1\omega}$-language admits infinitary conjunctions $\bigwedge\!\!\bigwedge$, and therefore instead of a set diag($\mathcal{M}$), a single formula $\bigwedge\!\!\bigwedge_{A \in \text{diag}(\mathcal{M})} A$ suffices for model $\mathcal{M}$'s description over $L_{\omega_1\omega}$. An analogue of that theorem might be found in Goldblatt's (1982, p. 174) Corollary 3.7.3; we spare a comment on Goldblatt's approach for the final discussion. Let us note also that the presence of ν, though convenient, is not essential for the proof of Scott's theorem, cf. (Passy, 1984).

Following (Parikh, 1981), with some change in the notation, we set (the sub-script L is for 'local'):

$\mathcal{M} \models \mathcal{N}$, if $\forall A\ (\mathcal{N} \models A$ only if $\mathcal{M} \models A)$,

$\mathcal{M} \models_L \mathcal{N}$, if $\forall x_{x \in M}\, \exists y_{y \in N}\, \forall A\ (\mathcal{M}, x \models A$ iff $\mathcal{N}, y \models A)$,

$\mathcal{M} \models\!\dashv_L \mathcal{N}$, if $\mathcal{M} \models_L \mathcal{N}$ and $\mathcal{N} \models_L \mathcal{M}$.

THEOREM 2.8. *For the combinatory language we have*:

$$\mathcal{M} \models_L \mathcal{N} \textit{ iff } \mathcal{M} \models \mathcal{N} \textit{ iff } \mathcal{N} \models \mathcal{M} \textit{ iff } \mathcal{M} \models\!\dashv \mathcal{N} \textit{ iff } \mathcal{M} = \mathcal{N}.$$

Proof. First we prove the only-if directions: for the first three of them, only the presence of v is essential, while the fourth comes from Corollary 2.7. Then the trivial implication from $\mathcal{M} = \mathcal{N}$ to $\mathcal{M} \models_L \mathcal{N}$ closes the proof-cycle, thus proving the if-directions. ▮

Such results do not exist in classical modal logic: Parikh (1981) notes that $\models\!\dashv$ does not imply $\models\!\dashv_L$, and also gives an example of two models $\mathcal{M}$ and $\mathcal{N}$, over the same universe, with $\mathcal{M} \models\!\dashv_L \mathcal{N}$, which are still non-isomorphic; this counter-example disproves the Correspondence theorem in the PDL case.

3. *Deductive System for CPDL*

We propose the following axiom schemes and rules.

(i) From PDL:

(Bool) All Boolean tautologies

($\Box_\alpha$) $[\alpha](A \to B) \to ([\alpha]A \to [\alpha]B)$

(;) $\langle \alpha; \beta \rangle A \leftrightarrow \langle \alpha \rangle \langle \beta \rangle A$

($\cup$) $\langle \alpha \cup \beta \rangle A \leftrightarrow \langle \alpha \rangle A \vee \langle \beta \rangle A$

(?) $\langle A? \rangle B \leftrightarrow A \wedge B$

(*) $\langle \alpha^* \rangle A \leftrightarrow A \vee \langle \alpha \rangle \langle \alpha^* \rangle A$

(ii) The Combinatory axioms:

(v1) $A \to \diamond A$

(v2) $\diamond \diamond A \to \diamond A$

(v3) $A \to \boxdot \diamond A$

(v4) $\langle \alpha \rangle A \to \diamond A$

—for the universe program v

(Σ1) $\diamond c$

(Σ2) $\diamond (c \wedge A) \to \boxdot (c \to A)$

—for the constants from Σ

(iii) The rules:

(MP) If $\vdash A$ and $\vdash A \to B$, then $\vdash B$.

(Ind) If $\vdash [\gamma][\alpha^k]A$, for all $k < \omega$, then $\vdash [\gamma][\alpha^*]A$.

(Cov) If $\vdash [\gamma] \neg c$, for all $c \in \Sigma$, then $\vdash [\gamma]0$.

(Nec) If $\vdash A$, then $\vdash \boxdot A$.

(iv) For CPDL$^\cap$ we add the axiom scheme

$$(\cap) \quad \langle \alpha \cap \beta \rangle c \leftrightarrow \langle \alpha \rangle c \wedge \langle \beta \rangle c$$

SOUNDNESS THEOREM 3.1. *If* $\vdash A$, *then* $\models A$.

Proof. Almost straightforward induction on $\vdash$. ▮

Now we show that the usual for PDL necessitation rule

$$(\mathrm{Nec}_\alpha) \quad \text{If } \vdash A, \text{ then } \vdash [\alpha]A$$

is admissible over CPDL, and that the Segerberg induction axiom

$$(\mathrm{ind})\, A \wedge [\alpha^*](A \rightarrow [\alpha]A) \rightarrow [\alpha^*]A$$

is a theorem of CPDL.

LEMMA 3.2. (Nec_α) *is an admissible rule, for each* α.

Proof. Immediate, by (Nec), (v4), and (MP). ▮

LEMMA 3.3. (ind) *is a theorem of CPDL.*

Proof. Take $\gamma =_{\mathrm{DF}} (A \wedge [\alpha^*](A \rightarrow [\alpha]A))?$, and apply (Ind), via induction on k. ▮

CONSERVATIVENESS THEOREM 3.4. *CPDL is a conservative extension of PDL (i.e. each PDL formula provable in CPDL is provable in PDL as well).*

Proof. The above two lemmata show that CPDL is indeed an extension of PDL. For conservativeness, let A be a PDL formula (i.e., v, $c \not\subset A$) with $\vdash A$. Then, by the Soundness theorem, $\models A$. The CPDL models are precisely the enrichments of the countable PDL models, hence A is valid in each of these last. Therefore, by the finite completeness of PDL, $\vdash_{\mathrm{PDL}} A$. ▮

Comments on v *and* Σ 3.5. A. For each $\alpha \in \Pi$, $[\alpha]$ is a K-modality (K, for Kripke, is the "minimal" modal logic determined by (Bool), $\Box_\alpha$, (Nec_α) and (MP)), and, moreover, $[\alpha]$ is a PDL-modality, whereas $\boxdot$ is even an S5. Therefore we shall take due dividents from deducibility in the sub-systems K, PDL, and S5 of CPDL; (K) will refer to formal manipulations in the sub-system K.

B. Axiom (v4) says that all programs agree with v, i.e., $R_\alpha \subseteq R_v$, for all α.

C. Axiom (Σ1), or equivalently $\Diamond 1$, says that each name names some state, and this axiom was the initial stimulus to introduce v in the language. Axiom ($\Sigma 2$), or equivalently $\langle c \rangle A \rightarrow [c] A$, says that any name c names only a single state, and more precisely, that the states namesakes are syntactically indescernible.

D. The ω-rule (Cov) intends to say (under the limitations of our finitary & propositional syntax) that each state is nominated, i.e., that the names *cover* the whole model, making χ surjective. By a contraposition, (Cov) hints that, if $\langle \gamma \rangle 1$ holds (at certain point), then there should be some c such that $\langle \gamma \rangle c$ holds (at the same point). This covering might be perfectly guaranteed by either an infinitary axiom $\vdash \bigvee_{c \in \Sigma} c$, or by a predicate one $\vdash (\exists c) c$ (see Chapter III). The prefixes $[\gamma]$ in (Ind) and (Cov) will be necessary for the proof of Deduction Lemma 4.13; thanks to the presence of ; and ? the compact prefix $[\gamma]$ serves instead of the usually employed *admissible forms* of (Goldblatt, 1982], in terms of which (Cov) would look like this:

$$\begin{aligned}&\text{If } \vdash [\alpha_1](A_1 \rightarrow [\alpha_2](A_2 \rightarrow \cdots [\alpha_k](A_k \rightarrow \neg c) \cdots)), \text{ for all } c \in \Sigma,\\ &\text{then } \vdash [\alpha_1](A_1 \rightarrow [\alpha_2](A_2 \rightarrow \cdots [\alpha_k](A_k \rightarrow 0) \cdots)).\end{aligned}$$

The rules (Ind) and (Cov) play a key role in our completeness proof, and we reserve some informal words on the ω-rules for the final discussion.

E. The reduction of the "global consequence" to the "consequence," discussed in Section 1, has general implications on the use of *finite* inference rules. In modal logics, we use the rule "from A infer B" twofold, to capture both:

(i) the global consequence: $\forall \mathcal{M}$ ($\mathcal{M} \models A$ only if $\mathcal{M} \models B$); and

(ii) say, the *universal* consequence: $\forall \mathcal{M}(\mathcal{M} \models A)$ only if $\forall \mathcal{M}$ ($\mathcal{M} \models B$).

Having ν one can therefore avoid the global use of the rule, postulating the axiom $\vdash \boxdot A \rightarrow \boxdot B$ instead. This had already been exemplified in Lemma 3.2, showing the rule "from A infer $[\alpha] A$" admissible. An illustration is also PCA's, $A\{\alpha\}B$, axiomatization: Floyd-Hoare rule of the type (see [Parikh 1981])

$$\text{“from } A_1\{\alpha_1\}B_1, ..., A_k\{\alpha_k\}B_k \text{ infer } A\{\alpha\}B\text{”}$$

simplifies to an ordinary axiom:

$$\vdash \bigwedge_{j=1}^{k} \boxdot(A_j \rightarrow [\alpha_j] B_j) \rightarrow \boxdot(A \rightarrow [\alpha] B).$$

F. The deductive system suggested for CPDL is perhaps not the most economic one: ($\nu 1$) and ($\nu 2$) follow from ($\nu 4$); the four axioms ($\nu 1$–4) and the single scheme $\langle\alpha\rangle A \rightarrow [\beta] \diamond A$ are interchangeable over CPDL; the axioms ($\Sigma 1, 2$) could be compacted as $\langle c \rangle A \leftrightarrow [c] A$, and our preferance in these cases is a matter of taste. Further, see Section 7, the ω-rule (Ind) and the axiom (ind) are interchangeable over the basic CPDL, where the rule (Cov) is even redudant. In these cases our choice is motivated by a long range axiomatic programme, keeping the system resistible to restrictions, extensions, and other mutations of the ground language.

Next we introduce the finitary rule

$$\text{(Cov*)} \qquad \text{If } \vdash [\gamma] \neg c, \text{ for some } c \not\propto \gamma, \text{ then } \vdash [\gamma] 0,$$

which is shown to be equivalent to (Cov), over CPDL. Before this equivalence is shown one lemma is needed,

Let us denote by

$$\frac{c_1 \cdots c_k}{d_1 \cdots d_k} A$$

(where $d_1, \ldots, d_k$ are *different* and $c_1, \ldots, c_k$ are arbitrary letters from Σ) the *uniform substitution* of $d_1, \ldots, d_k$ in A, by $c_1, \ldots, c_k$.

SYMMETRY LEMMA 3.6. *If* $\vdash A$, *then, for each substitution* σ, $\vdash \sigma A$.

Proof. (By induction on $\vdash$.)

If A is an axiom, then the claim is clear by inspection.

If A is deduced by some of the rules (MP), (Nec), or (Ind), then the result follows from IH and the definition of substitution, the respective rule being applied.

If A is deduced by (Cov), then $A = [\gamma] 0$, for some γ, and $\vdash [\gamma] \neg c$, for all c. Let

$$\sigma =_{\text{DF}} \frac{c_1 \cdots c_k}{d_1 \cdots d_k},$$

and let $d \not\propto \gamma$ and $d \notin \{c_1, \ldots, c_k, d_1, \ldots, d_k\}$. Let

$$\sigma_c =_{\text{DF}} \frac{c c_1 \cdots c_k}{d d_1 \cdots d_k}.$$

We have $\vdash [\gamma] \neg d$, hence by IH, $\vdash \sigma_c [\gamma] \neg d$; and $\sigma_c [\gamma] \neg d = [\sigma_c \gamma] \sigma_c \neg d = [\sigma\gamma] \neg c$. So, $\vdash [\sigma\gamma] \neg c$, which is the case for all c. By (Cov), $\vdash [\sigma\gamma] 0$, i.e. $\vdash \sigma A$. ∎

LEMMA 3.7. *The rule* (Cov*) *is admissible.* (*This was pointed to us by Dimiter Skordev.*)

Proof. Let $\vdash [\gamma] \neg c$, for some $c \not\propto \gamma$. Taking

$$\sigma = \frac{d}{c},$$

by the Symmetry lemma 3.6, we obtain that $\vdash [\gamma] \neg d$, which is the case for arbitrary d. Hence, by (Cov), $\vdash [\gamma]0$. ▮

LEMMA 3.8. *The rules* (Cov*) *and* (Cov) *are interchangeable for CPDL.*

Proof. The less trivial implication is Lemma 3.7. ▮

Remark on the (Cov)*-type Infinitary Rules* 3.9. Having the equivalence between (Cov) and the finitary (Cov*), we say that (Cov) is a *quasi-ω-rule for* CPDL. Obviously, (Cov) will be a quasi-ω-rule for each extension of CPDL respecting the Symmetry lemma. Let us take an arbitrary ω-rule of the type

(R) If $\vdash [\gamma] A(c_1, ..., c_k)$, for each substitution of $c_1, ..., c_k$ in A, then $\vdash [\gamma] B$.

(Obviously, (Ind) is not of that type.) Let us define (R)'s finitary version

(R*) If $\vdash [\gamma] A(c_1, ..., c_k)$, for some $c_1, ..., c_k \not\propto \gamma$, then $\vdash [\gamma] B$.

A trivial generalization of Lemma 3.8 says that, if some extension $\mathscr{E}$ of CPDL is *symmetric*, i.e., if it respects the Symmetry lemma 3.6, than (R) and (R*) are interchangeable for $\mathscr{E}$, i.e., that (R) is a quasi-ω-rule for $\mathscr{E}$. By $*\mathscr{E}$ we shall denote the version of $\mathscr{E}$ in which each such (R) is replaced by (R*), not necessary meaning that $\mathscr{E} = *\mathscr{E}$.

The following facts will be frequently used in the sequel.

Exercise 3.10.

(i) $\vdash \langle\cdot\rangle(c \wedge A) \leftrightarrow \langle \mathrm{c} \rangle A$

$\vdash [\cdot](c \rightarrow A) \leftrightarrow [\mathrm{c}] A$

(ii) $\vdash \langle \mathrm{c} \rangle A \leftrightarrow [\mathrm{c}] A$

(iii) $\vdash \langle \mathrm{c} \rangle (A \wedge B) \leftrightarrow \langle \mathrm{c} \rangle A \wedge \langle \mathrm{c} \rangle B$

(iv) $\vdash \langle \mathrm{c} \rangle A \rightarrow [\alpha](c \rightarrow A)$

(v) $\vdash \langle \mathrm{c} \rangle \langle\cdot\rangle d.$

LEMMA 3.11. $\vdash \langle\alpha\rangle(c \wedge A) \leftrightarrow \langle\alpha\rangle c \wedge \langle c\rangle A$.

Proof. (We recall that (K) denotes provability in the minimal modal logic K.)

$(\rightarrow)$ by (K) and by $(\nu 4)$.

$(\leftarrow)$ $\vdash \langle\alpha\rangle c \wedge \langle c\rangle A \rightarrow \langle\alpha; c?\rangle 1 \wedge \boxdot(c \rightarrow A)$, by (;), (?), and $(\nu 4)$

$\rightarrow \langle\alpha; c?\rangle 1 \wedge [\alpha](c \rightarrow A)$, by $(\nu 4)$

$\rightarrow \langle\alpha; c?\rangle 1 \wedge [\alpha; c?]A$, by (?) and (;)

$\rightarrow \langle\alpha; c?\rangle A$, by (K)

$\rightarrow \langle\alpha\rangle(c \wedge A)$, by (;) and (?). ▮

COROLLARY 3.12. $\vdash \langle e\rangle\langle\alpha\rangle c \wedge \langle c\rangle A \rightarrow \langle e\rangle\langle\alpha\rangle A$.

Proof. Substitute $\hat{e};\alpha$ for α. ▮

DEFINITION 3.13. The formula A is *closed*, if for some B, $\vdash A \leftrightarrow \diamond B$. Let $A^{\sim}$ range over the closed formulae and their Boolean combinations (which also turn out to be closed).

For instance, $\langle c\rangle B$, $[c]B$, $\boxdot B$, are closed.

AN S5 EXERCISE 3.14. $\vdash A^{\sim} \leftrightarrow \boxdot A^{\sim}$, $\vdash A^{\sim} \leftrightarrow \diamond A^{\sim}$, $\vdash \diamond A^{\sim} \leftrightarrow \boxdot A^{\sim}$.

EXERCISE 3.15. $\vdash [\gamma]A^{\sim} \leftrightarrow A^{\sim} \vee [\gamma]0$.

Proof.

$(\leftarrow)$ $\vdash (A^{\sim} \rightarrow [\gamma]A^{\sim}) \wedge ([\gamma]0 \rightarrow [\gamma]A^{\sim})$, by Exercise 3.14, $(\nu 4)$ and (K).

$(\rightarrow)$ $\vdash [\gamma]A^{\sim} \wedge \neg[\gamma]0 \rightarrow \langle\gamma\rangle A^{\sim}$, by (K).

$\rightarrow \diamond A^{\sim}$, by $(\nu 4)$

$\rightarrow A^{\sim}$, by Exercise 3.14. ▮

COROLLARY 3.16. $\vdash A^{\sim} \leftrightarrow [c]A^{\sim}$.

Reflexions on Intersection 3.17. Let us turn back to the initial tri-modal case discussed in the Introduction, with modalities $\langle\alpha \cap \beta\rangle$, $\langle\alpha\rangle$, $\langle\beta\rangle$, free of * and the other operations. In this case we shall have an admissible-forms-analogue of (Cov); see Comment 3.5.D. Since the axiom scheme $(\cap)$ $\vdash \langle\alpha \cap \beta\rangle c \leftrightarrow \langle\alpha\rangle c \wedge \langle\beta\rangle c$ respects the Symmetry lemma, this only ω-rule vanishes as well, leaving behind a nice finitary axiomatics for the intersection. Therefore, we have to have $\langle\alpha\cap\beta\rangle A\rightarrow\langle\alpha\rangle A\wedge\langle\beta\rangle A$ provable. Indeed we have

PROPOSITION. $CPDL^{\cap}\vdash(\langle\alpha\cap\beta\rangle A\rightarrow\langle\alpha\rangle A\wedge\langle\beta\rangle A)$.

Proof. Through (Cov), in about 10 steps. Omitted. (See the proof of Lemma II.1.1.) ▮

We shall give an axiom for $\cup$ that is dual to ($\cap$), thus shading the stimulating modal contrast between $\cup$ and $\cap$.

PROPOSITION. *The axiom* ($\cup'$) $\langle\alpha\cup\beta\rangle c\leftrightarrow\langle\alpha\rangle c\vee\langle\beta\rangle c$ *is interchangeable, over CPDL, with the union axiom* ($\cup$) $\langle\alpha\cup\beta\rangle A\leftrightarrow\langle\alpha\rangle A\vee\langle\beta\rangle A$.

Proof. Through (Cov), in about 20 steps. Omitted. (See the proof of Lemma II.1.1.) ▮

4. *Proof Theory*

In this section we investigate the syntactic ω-analogues of model and state—the notions of "logic" and "theory." The names and v, on the one hand, import some fresh dependences between these two notions, and on the other hand, do not damage the classical results: Deduction lemma, Separation lemma, Lindenbaum lemma. The technique exploited will be used several times in the sequel—when infinitary rules are dealt with. This technique is transferred from (Tinchev and Vakarelov, 1983), and it probably originates from the Q-filters machinery of (Rasiowa and Sikorski, 1963), or later from (Goldblatt, 1982).

DEFINITION 4.1. A *simple extension of CPDL*, or *logic* (over CPDL) is any set of CPDL formulae L such that:

(a) L contains all axioms of CPDL; and

(b) L is closed under (MP), (Ind), (Cov), and (Nec).

DEFINITION 4.2. Where L is a logic, an *L-theory* is any set $T\subseteq\Phi$ such that:

(a) $L\subseteq T$; and

(b) T is closed under (MP), (Ind), and (Cov),

By *deductive set* we mean logic or theory. The letters L and T will range over logics and theories, respectively; $L\vdash A$, $\vdash_L A$, and $A\in L$ are synonyms.

DEFINITION 4.3. A deductive set Q is *consistent*, if $0 \notin Q$.

DEFINITION 4.4. (i) T is *maximal*, if $\forall A$ (either $A \in T$ or $\neg A \in T$)
(ii) L is *maximal*, if $\forall A^{\sim}$ (either $A^{\sim} \in L$ or $\neg A^{\sim} \in L$).

Obviously, each logic is a theory; and the maximal deductive sets are consistent ones. The next five lemmas are easy exercises.

LEMMA 4.5. *If L is consistent, then, for all c, $\neg c \notin L$.*

LEMMA 4.6. *If T is maximal, then, for some c, $c \in T$.*

Notation 4.7. $L^c =_{\mathrm{DF}} \{A / \langle c \rangle A \in L\}$; $\mathscr{L}_T =_{\mathrm{DF}} \{A / \boxdot A \in T\}$.

LEMMA 4.8. *If L is a (maximal) logic, then L^c is a (maximal) theory, and $L = \bigcap \{L^c / c \in \Sigma\}$.*

LEMMA 4.9. *If T is a (maximal) theory, then $\mathscr{L}_{\mathrm{T}}$ is a (maximal) logic, and $\mathscr{L}_{\mathrm{T}}$ is the greatest logic included in T.*

LEMMA 4.10. (i) *If $c \in T$, then $(\mathscr{L}_T)^c = T$.*
(ii) *For each c, $\mathscr{L}_{(L^c)} = L$.*

DEFINITION 4.11. By $\mathrm{log}(\Gamma, A)$ (respectively, $\mathrm{th}(\Gamma, A)$ we denote the least logic (theory) containing $\Gamma \cup \{A\}$.

Obviously, $\mathrm{th}(\Gamma, A) \subseteq \mathrm{log}(\Gamma, A)$.

LEMMA 4.12. *If $\forall B$ ($B \in \Gamma \cup \{A\}$ implies $\boxdot B \in \mathrm{th}(\Gamma, A)$), then $\mathrm{th}(\Gamma, A)$ is a logic.*

Proof. Straightforward induction on deducibility in $\mathrm{th}(\Gamma, A)$. ▮

COROLLARY. $\mathrm{th}(L, A^{\sim}) = \mathrm{log}(L, A^{\sim})$.

DEDUCTION LEMMA FOR THEORIES 4.13. *$A \to B \in T$ iff $B \in \mathrm{th}(T, A)$.*

Proof. The "interesting" part is (if). Let $B \in \mathrm{th}(T, A)$ and $T_0 =_{\mathrm{DF}} \{D / A \to D \in T\}$. T_0 is an L-theory: Obviously, $L \subseteq T \subseteq T_0$ and T_0 is (MP)-closed. The (Ind)-, respectively, (Cov)-closeness of T_0 follows immediately from that of T, by the axiom (?). (At that point we actually make use of the prefixes $[\gamma]$ in (Ind) and (Cov).) So, the theory T_0 contains $T \cup \{A\}$, and hence $B \in \mathrm{th}(T, A) \subseteq T_0$. By the definition of T_0, $A \to B \in T$. ▮

DEDUCTION LEMMA FOR LOGICS 4.14. $A^{\sim} \to B \in L$ *iff* $B \in \log(L, A^{\sim})$.

Proof. By the Deduction lemma for theories and the Corollary to Lemma 4.12. ▌

SEPARATION LEMMA FOR THEORIES 4.15. *Let $A \notin T$. Then there exists a maximal theory T^* such that $T \subseteq T^*$ & $A \notin T^*$.*

Proof. Let $T_0 =_{\mathrm{DF}} \mathrm{th}(T, \neg A)$. By the Deduction lemma for theories, T_0 is consistent. Let $B_0, B_1, B_2, \ldots$ be an enumeration of Φ. (Here we use the countability of CPDL's language.) By induction on n, we shall construct a chain $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ of consistent theories. Their union will yield the required T^*. (The main concern will be to ensure that, if $\langle\gamma\rangle 1$ (resp., $\langle\gamma\rangle\langle\alpha^*\rangle A$) occurs in the chain, then for some c (resp., for some k), $\langle\gamma\rangle c$ (resp., $\langle\gamma\rangle\langle\alpha^k\rangle A$) should occur in the chain as well.)

IH: Let T_n be defined as a consistent theory.

Let, for short, $D =_{\mathrm{DF}} B_n$, and (Ω) stand for (Ind) or (Cov).

(A) If $\mathrm{th}(T_n, D)$ is consistent, then $T_{n+1} =_{\mathrm{DF}} \mathrm{th}(T_n, D)$ is evidently consistent.

(B) If $\mathrm{th}(T_n, D)$ is not consistent, then, by the Deduction lemma for theories, $\neg D \in T_n$.

(B1) Let B be not of the type "consequence of (Ω)," i.e., $D \neq [\gamma]0$ and $D \neq [\gamma][\alpha^*]A$. Then $T_{n+1} =_{\mathrm{DF}} T_n$ is consistent.

(B2) Let D be of the type "consequence of (Ω)"—i.e., $D = [\gamma]0$ or $D = [\gamma][\alpha^*]A$—with $\{D_j/j < \omega\}$ being the premises of that (Ω).

If $\forall j(D_j \in T_n)$, then the (Ω)-closeness of T_n and $\neg D \in T_n$ would imply inconsistency of T_n, which, by IH, is not the case. So, $\exists q\ (D_q \notin T_n)$, and by the Deduction lemma for theories, $T_{n+1} =_{\mathrm{DF}} \mathrm{th}(T_n, \neg D_q)$ is consistent. This concludes the definition of T_{n+1}.

Let $T^* =_{\mathrm{DF}} \bigcup\{T_n/n < \omega\}$. We have:

(a) $L \subseteq T \subseteq T_0 \subseteq T^*$;

(b) T^* is (MP)-closed;

(c) $A \notin T^*$: by $\forall k\ (\neg A \in T_0 \subseteq T_k$ & T_k is consistent);

(d) $0 \notin T^*$: by (b) and (c);

(e) $\forall B\ (B \in T^*$ or $\neg B \in T^*)$: this comes from cases (A) and (B) above;

(f) T^* is (Ω)-closed: let D_j, for $j < \omega$, be the premises and D, the consequence of (Ω). Let $\forall j\ (D_j \in T^*)$, and let D be B_n. Suppose $B_n \notin T^*$, then we should have the case (B2) and some $q < \omega$ with $\neg D_q \in T_{n+1} \subseteq T^*$, i.e., $\neg D_q \in T^*$ & $D_q \in T^*$—a contradiction with (d).

Thus (a–f) state that T^* is a maximal L-theory and $T \subseteq T^*$ and $A \notin T^*$. ▮

SEPARATION LEMMA FOR LOGICS 4.16. *Let $A \notin L$. Then there exists a maximal logic L^* such that $L \subseteq L^*$ & $A \notin L^*$.*

Proof. We have that $A \notin L$, whence $\boxdot A \notin L$, and that L is a theory as well. By the Separation lemma for theories, there is some maximal L-theory T with $\boxdot A \notin T$ (& $L \subseteq T$). By Lemma 4.9, $\mathscr{L}_T$ is the required L^*. ▮

LINDENBAUM LEMMATA 4.17 (Corollaries to the Separation lemmata). *Each consistent theory (logic) can be extended to a maximal one.*

DEFINITION 4.18. A logic is *categorical*, if it has no more than one model (up to isomorphism).

LEMMA 4.19. *If Q is a maximal deductive set, then Q contains a full set, cf. Definition* 2.2.

Proof. This is the full set over $\{A \in Q/A$ is Elementary Formula$\}$. ▮

THEOREM 4.20. *If a logic contains a full set, then it is categorical.*

Proof. By Scott's Theorem 2.6. ▮

THEOREM 4.21. *Each maximal logic is categorical.*

Proof. By Lemma 4.19 and Theorem 4.20. ▮

Now we are ready to give a slightly unorthodox completeness proof (cf. Section 0.6) for modal logic: we take a disprovable formula A, and according to the Separation lemma, blow its negation $\neg A$ up to a maximal theory T; T should contain both a full set Γ (Lemma 4.19), and some name c_0 (Lemma 4.6). Then Γ determines a model $\mathscr{M} = \mathrm{mod}(\Gamma)$, cf. Section 2. Now it suffices to prove that each formula from T is satisfied at the state of $\mathscr{M}$ determined by c_0, i.e., that $\mathscr{M}, |c_0| \models A$, for each $A \in T$. We omit this trifle, with a reference to Note II.1.7 below.

In the next section we complete a slightly different version of the completeness proof, which is not explicitly dependent on the results from Section 2, and which will be the base for the remaining completeness results throughout the paper.

5. *Completeness Theorem for Simple Extensions of CPDL and of CPDL*$^\cap$

TRUTH LEMMA 5.1. *Let T be a maximal L-theory. Then:*

(1) $\Diamond_c A \in T$ *iff* $\Box_c A \in T$

(2) $\Diamond_c \neg A \in T$ *iff* $\Diamond_c A \notin T$

(3) $\Diamond_c (A \vee B) \in T$ *iff* $\Diamond_c A \in T$ *or* $\Diamond_c B \in T$

(4) $\Diamond_c \langle \alpha \rangle A \in T$ *iff* $\exists e(\Diamond_c \langle \alpha \rangle e \in T \,\&\, \Diamond_e A \in T)$

(5) $\Diamond_c \diamond d \in T$

(6) $\Diamond_c \langle \alpha\beta \rangle d \in T$ *iff* $\exists e(\Diamond_c \langle \alpha \rangle e \in T \,\&\, \Diamond_e \langle \beta \rangle d \in T)$

(7) $\Diamond_c \langle \alpha \cup \beta \rangle d \in T$ *iff* $\Diamond_c \langle \alpha \rangle d \in T$ *or* $\Diamond_c \langle \beta \rangle d \in T$

(8) $\Diamond_c \langle \alpha^* \rangle d \in T$ *iff* $\exists k_{k \geqslant 0}(\Diamond_c \langle \alpha^k \rangle d \in T)$

(9) $\Diamond_c \langle A? \rangle d \in T$ *iff* $\Diamond_c d \in T \,\&\, \Diamond_d A \in T$.

For CPDL$^\cap$ *we add*:

(10) $\Diamond_c \langle \alpha \cap \beta \rangle d \in T$ *iff* $\Diamond_c \langle \alpha \rangle d \in T \,\&\, \Diamond_c \langle \beta \rangle d \in T$.

Proof. (1) follows from Exercise 3.10(ii).

(2) follows from (1) and the definition of maximal theory.

(3), in the interesting 'only if' direction, follows from (2) and (K).

(4, if) follows from Corollary 3.12.

(4, only if) Let $\Diamond_c \langle \alpha \rangle A \in T$. Suppose, for contradiction, that $\forall e$ $(\langle \hat{c}\alpha \rangle e \notin T$ or $\Diamond_e A \notin T)$. Then, by the maximality of T, $[\hat{c}\alpha] \neg e \in T$ or $\Diamond_e \neg A \in T$, for all e. By (K) and Exercise 3.10(iv), we have that in either case, $[\hat{c}\alpha](A \to \neg e) \in T$, for all e. Now (Cov) leads to a contradiction.

(5) is Exercise 3.10(v).

(6) follows immediately from (4) with $A = \langle \beta \rangle d$, and axiom (;).

(7) follows from (3) and axiom ($\cup$).

(8, if) By induction on k, we prove that $\vdash \langle \alpha^k \rangle A \to \langle \alpha^* \rangle A$, hence we have that $\vdash \Diamond_c \langle \alpha^k \rangle d \to \Diamond_c \langle \alpha^* \rangle d$ which suffices.

(8, only if) Let $\Diamond_c \langle \alpha^* \rangle d \in T$, and suppose that $\forall k$ $(\Diamond_c \langle \alpha^k \rangle d \notin T)$. Then, by (2) and Exercise 3.10(ii), $\forall k([\hat{c}][\alpha^k] \neg d \in T)$, and now the (Ind)-closeness of T yields $[\hat{c}][\alpha^*] \neg d \in T$, which contradicts $\langle \hat{c} \rangle \langle \alpha^* \rangle d \in T$.

(9) follows from axiom (?) and Lemma 3.11.

(10) follows from axiom ($\cap$) and Exercise 3.10(iii). ∎

Note. (Cov) was essentially used in (4) and (6), and (Ind) in (8).

We recall that $\mathcal{M}$ is a *model for L*, or *L-model*, if $\mathcal{M} \models L$, and that $\models_L$ stands for validity in all L-models.

COMPLETENESS THEOREM, FIRST FORM 5.2. *If L is consistent, then L has a model.*

COMPLETENESS THEOREM, SECOND FORM 5.3. *If* $\models_L A$, *then* $\vdash_L A$.

LEMMA 5.4. *The two forms of the completeness theorem are equivalent.*

Proof. First form implies Second form: Let $\nvdash_L A$, i.e., $A \notin L$, and so $\Box A \notin L$. By the Deduction lemma 4.14, the logic $\log(L, \Diamond \neg A)$ is consistent, and by the First form, it has a model $\mathcal{M}$. Since the L-model $\mathcal{M} \models \Diamond \neg A$, i.e., $\mathcal{M} \not\models A$, $\not\models_L A$. Second form implies First form: Let L have no models. So $\models_L 0$, and by the Second form, $\vdash_L 0$, i.e., L is inconsistent. ∎

Proof of the Completeness Theorem, First Form. The logic L is a consistent L-theory. By the Lindenbaum lemma 4.17, there exists a maximal L-theory T.

Let $c \sim d$ stand for $\langle c \rangle d \in T$, and $|c| =_{\text{DF}} \{d/d \in \Sigma \;\&\; c \sim d\}$.

Let $M =_{\text{DF}} \Sigma_{/\sim}$, $\chi(c) =_{\text{DF}} |c|$ (χ is obviously surjective on M), $V(A) =_{\text{DF}} \{|c|/\langle c \rangle A \in T\}$, $R_\alpha =_{\text{DF}} \{(|c|, |d|)/\langle c \rangle \langle \alpha \rangle d \in T\}$. An inspection verifies the correctness of this factorization. By the Truth lemma 5.1 we obtain that $\mathcal{M} = (M, R, \chi, V)$ is indeed a model, which we call the *canonical model for L* (generated by T). Indeed, $\mathcal{M}$ is an L-model: Let $A \in L$. Then $[c] A \in L \subseteq T$, i.e., $\langle c \rangle A \in T$. By the definition of V, $\mathcal{M}, |c| \models A$, which is the case for each $c \in \Sigma$. So $\mathcal{M} \models A$. ∎

6. *Downward Löwenheim–Skolem Arguments*

The completeness theorem just proven justifies our choice of "χ-surjective," whence countable models, which is, on the one hand, a restrictive option, but, on the other, a common sin today (cf. (Harel, 1984)). Another vindication for this choice—a version of the downward Löwenheim–Skolem theorem—will be presented below.

Let us for a moment drop the restriction in Definition 2.1 for countability of the models, hence for surjectivity of χ, and call the non-surjective models thus obtained *nsj-models*. "Non-surjective" means that χ is not necessary surjective; nsj-validity, denoted $\models_{\text{NSJ}}$, means validity in nsj-models.

LEMMA 6.1. *Let* $\mathcal{M} = (M, R, \chi, V)$ *be an nsj-model; let* $s_0 \in M$ *and* $c_0 \in \Sigma$. *Then there exists an nsj-model* $\mathcal{M}^+ = (M, R^+, \chi^+, V^+)$ *such that*

$\chi^+(c_0)=s_0$, and for each A and α, if $c_0 \not\propto A$, α, then $R^+_\alpha = R_\alpha$ & $V^+(A)=V(A)$.

Proof. We define $\chi^+(c_0)=s_0$, and, for $a \in \Pi_0$, $p \in \Phi_0$, $c \in \Sigma \setminus \{c_0\}$,

$$R^+_a = R_a, \qquad V^+(p)=V(p), \qquad \chi^+(c)=\chi(c).$$

Then we extend R^+ and V^+ inductively on Π and Φ and obtain the required $\mathscr{M}^+$. ▮

COROLLARY 6.2. *Let, in the above notation, $sR_\alpha s_0$ & $c_0 \not\propto \alpha$. Then we have $sR^+_\alpha s_0$ and $\mathscr{M}^+$, $s_0 \models c_0$. Therefore $\mathscr{M}^+$, $s \models \langle\alpha\rangle c_0$.*

SOUNDNESS THEOREM FOR nsj-MODELS 6.3. *If $\vdash A$, then $\models_{\text{nsj}} A$.*

Proof. (By induction on $\vdash$.) All axioms of CPDL are obviously nsj-valid. For all the rules but (Cov) we have more than is required: if the premises of such a rule are valid in a fixed nsj-model, then the consequence will be valid in the same model. Hence nsj-validity of the premises implies nsj-validity of the consequence.

Let the consequence $[\alpha]0$ of (Cov) be not nsj-valid. Hence for some nsj-model $\mathscr{M}=(M, R, \chi, V)$, and $s \in M$, $\mathscr{M}$, $s \models \langle\alpha\rangle 1$; i.e., there is some $s_0 \in M$ with $sR_\alpha s_0$. Let $c_0 \not\propto \alpha$. Then, in the notation of Corollary 6.2, $\mathscr{M}^+$, $s \not\models [\alpha]\neg c_0$. So not all the premises of this (Cov) are nsj-valid. ▮

Since nsj-validity implies standard validity, the Completeness theorem, Second form, 5.3, combined with the above theorem proves

DOWNWARD LÖWENHEIM–SKOLEM THEOREM 6.4. (i) *If A has an uncountable model, then A has a model as well;*

(ii) *If A is "uncountably" satisfiable, then A is satisfiable as well.*

Anyway, the nsj-model, though reliable for modelling a single formula, is not an adequate notion for our (Cov)-based incompact proof theory: there is a set of formulae which does have an nsj-model, and which is nevertheless inconsistent, and therefore with no standard model. Such is the set

$$\{\langle c_0?; a\rangle 1\} \cup \{[c_0?; a]\neg c_k / k < \omega\},$$

which is modelled on each nsj-model $\mathscr{M}_0$, in which the state $\chi(c_0)$ has some a-successor, but no named a-successors.

Another counter-example is presented by the failing upward Löwenheim–Skolem theorem (or better say failing big model property): the formula $\boxdot(c_1 \vee \cdots \vee c_k)$ is obviously satisfied in models $\mathscr{M}$ with $\text{card}(\mathscr{M}) \leqslant k$, and in no infinite model.

7. *Finitary Axiomatization and Decidability Results for CPDL*

Gargov (1985), following Segerberg's (1977) filtration method, establishes a finitary axiomatization, the finite model property (fmp), and hence decidability of CPDL. A review of his results follows.

DEFINITION 7.1. Let Finitary CPDL, $\mathscr{F}$CPDL, denote the system obtained from CPDL by dropping the two ω-rules (Ind) and (Cov), and adding Segerberg's axiom (ind): $A \wedge [\alpha^*](A \to [\alpha]A) \to [\alpha^*]A$. Let $\vdash_{\mathrm{F}}$ denote provability in $\mathscr{F}$CPDL.

Obviously, the theorems of $\mathscr{F}$CPDL form an r.e. set.

THEOREM 7.2. *If* $\vdash_{\mathrm{F}} A$, *then* $\vdash A$.

Proof. The only non-trivial step is in Lemma 3.3 above. ∎

THEOREM 7.3 (Gargov, 1985). *If* $\not\vdash_F A$, *then, for some* finite *model* $\mathscr{M}$, $\mathscr{M} \not\models A$.

Proof. Through filtration, cf. (Segerberg, 1977). ∎

So we have that the non-theorems of $\mathscr{F}$CPDL also form an r.e. set. Hence:

COROLLARY 7.4. *$\mathscr{F}$CPDL is decidable.*

The decision procedure for $\mathscr{F}$CPDL has exactly the same complexity as the one for PDL: deterministic bounds exponential in the size of the formula.

THEOREM 7.5. *If* $\vdash A$, *then* $\vdash_F A$.

Proof. By the Soundness of CPDL and the Completeness theorem 7.3 for $\mathscr{F}$CPDL. ∎

COROLLARY 7.6. $\vdash A$ *iff* $\vdash_F A$.

COROLLARY 7.7. *CPDL is decidable* (*in the same exponential bounds as $\mathscr{F}$CPDL and PDL*).

Corollary 7.6 indicates two curious finitistic points for the basic system CPDL: that the rule (Ind) is reducible to (ind) and that (Cov) is not only quasi-ω, but even redundant. We leave most of the comments we are able to make on these points to Section II.6 and the final Discussion. However, we hasten to show the negative answer (given by Goranko) to a question suggested by Corollary 7.6: "is (Cov) always redundant?"

THEOREM 7.8. *There are extensions of CPDL for which the rule* (Cov) *is not redundant.*

Proof. Let, for a fixed program letter a,

$$\mathrm{CPDL}_0 =_{\mathrm{DF}} (\mathrm{CPDL\ without\ (Cov)}) + \{\vdash [c_0?; a] \neg c / c \in \Sigma\},$$

and let $\mathscr{M}_0$ be an nsj-model, as described at the end of the previous section, in which $\chi(c_0)$ has some a-successor, but no named a-successor. By induction on provability in CPDL_0, one obtains that

$$\mathrm{CPDL}_0 \vdash A \qquad \text{only if} \quad \mathscr{M}_0 \models A.$$

Hence, since $\mathscr{M}_0 \not\models [c_0?; a]0$, we have that $\mathrm{CPDL}_0 \nvdash [c_0?; a]0$, and therefore the possible addition of (Cov) over CPDL_0 would bring some new theorems. ∎

As known since (Meyer *et al.* 1981) and (Harel, 1983), the decidability of the *global consequence* and the *consequence* problems for PDL (i.e., the problems of whether A is a consequence—global or not—of a set of formulae Γ, cf. Section 1) are both Π^1_1-hard, and precisely Π^1_1-complete. As noted by Harel (1984), this result, being negative in nature, holds for extensions of PDL too, and so it holds for CPDL and its extensions.

THEOREM 7.9. *The decidability of both the consequence and the global consequence problems for CPDL is* Π^1_1*-complete.*

Convention 7.10. Given a logic L, we shall denote by Finitary L, $\mathscr{F} L$, the system obtained from L, by the prescription of Definition 7.1 above. Extending the stipulation from 3.9, by $\mathscr{F}{*L}$ we shall denote the $\mathscr{F}$- version of $*L$; so, if (Ind) and (Cov) are the only ω-rules, $\mathscr{F}{*L} = \mathscr{F} L + (\mathrm{Cov}^*)$.

This section presents a case of results not extendable to the intersectioned language: the filtration fails for $\mathscr{F}\,\mathrm{CPDL}^{\cap}$. We leave the decidability questions for $\mathrm{CPDL}^{\cap}$, most of which are negative or open, to Section II.6.

Chapter II: Extensions of CPDL, or Applications of the Method

In this chapter, several types of extensions of CPDL will be considered, completeness being a central point. In Section 1, three illustrations for *simple extensions* are discussed. One is the logic of deterministic atomic programs CDPDL, and the other is the logic of the so-called n-models, namely models whose cardinality is limited by the natural number n. The third example violates the classical modal canons, presenting, for each model $\mathscr{M}$, a logic $\mathrm{CPDL}^{\mathscr{M}}$ whose only model is $\mathscr{M}$.

Section 2 is motivated by some *definitional extensions*, which are traditionally in the scope of dynamic and modal logics: intersection, union, complementation, converse, inclusion, and equality of programs. Each of these operations θ is definable through some PC formula F, and the question is how to axiomatize θ, given F. We show in Section 2.1 that if F is modally expressible, then the expressive formula immediately yields an axiomatization of θ. Most of the familiar operations are modally expressible, but nevertheless, the expressible formulae do not even include all $\forall^*$- and $\exists^*$-PC formulae. In Section 2.2, we show that the addition of one quasi-ω-rule increases the axiomatic power of CPDL up to axiomatization of each operator definable through a $\forall^*$- or $\exists^*$-prefixed-expressible-formula, which readily covers all $\forall^*$- and $\exists^*$-formulae. The ultimate answer to that question is given in the next chapter, where the language of Quantificational CPDL, CDL, provides expressiveness, and a fortiori axiomatization, of all PC-definable operators, and thus ousts all quasi-ω-rules.

Section 3 deals with some extensions treating infinity. Considered are degrees of program non-determinism, including the predicate $\nabla\alpha$, which is true at points with infinitely many immediate α-successors; the logics of finite and infinite models (controlled by the axioms $\neg\nabla\nu$ and $\nabla\nu$); the traditional Polish iteration quantifier $\Pi\alpha$, which is true at points followed by arbitrary long paths of α-successors; axiomatization *a la* Streett of the familiar repeating $\Delta\alpha$, true at points followed by infinite paths of α-successors. These last, $\Pi\alpha$, $\nabla\alpha$, $\Delta\alpha$, and α^* are finally taken in a joint context to formalize Brouwer and König's lemma: $\Pi\alpha \rightarrow \Delta\alpha \vee \langle\alpha^*\rangle\nabla\alpha$.

In Section 4, two exotic extensions of CPDL are proposed, admitting a choice function in the models. One is the uniform selector operator ρ, where $\rho\alpha$ chooses a functional branch of the relation α, with the same domain. Another is the well-ordering predicate $\leqslant$, whose axiomatization utilizes ρ, asking $\rho\alpha$, in state s, to select the least element of $R_\alpha(s)$.

Section 5 initiates another project. Suggested is a development of the propositional dynamic logic—which is both monadic (of one-argument programs, $\alpha(x)$) and unary (of one-place modalities, $[\alpha]p$)—into a polyadic ($\alpha = \alpha(x_1, ..., x_k)$) and multi-ary ($[\alpha] = [\alpha](p_1, ..., p_j)$) theory. This suggestion is grounded on two representative examples. One is the extension with diadic function, coding pairs of possible worlds. The second is in a curious solution to an old question. The question was posed by Segerberg (1973), appealing to a simple axiomatization of two-dimensional modal logic (whose universes are Cartesian squares), the answer being given by A. Petkov (1987), whose elegant method applies to even all Cartesian degrees. These examples back the combinatory modal theory in its claims to be chosen as a basis for a "general theory of binary propositional connectives," in the sense of [Došen, 1986].

In Section 6, we discuss the role of the ω-rule and some decidability questions.

Warning. The results in this and the next chapters are presented a bit sketchily. By Completeness we shall mean the First form of the Completeness theorem. Each particular system in this and the next chapter will be an extension of CPDL, and so we agree to take it for granted that its syntax, semantics, deductive system, and even Truth lemma will extend, properly or not, those of CPDL. Those readers who feel some lack of motivation and argumentation for the investigations and the results in this chapter (in particular those starting to read the paper from this chapter) are advised to consult Chapter 0 and the final Discussion, while Chapter 1 is a source for restoration of the missing completeness proofs, which, as a rule, will be mechanical extensions of the completeness proof for CPDL.

1. *Three Simple Extensions*

1.1. *Determinism: CDPDL*

We enlarge CPDL's deductive system with the scheme

$$(\mathrm{det}) \vdash \langle a \rangle c \rightarrow [a]c, \qquad \text{for} \quad c \in \Sigma,$$

for all $a \in \Pi_0$ and call the extension Combinatory DPDL. By (DET) we shall denote the usual model "functionality" scheme:

$$(\mathrm{DET}) \vdash \langle a \rangle A \rightarrow [a]A, \qquad \text{for all } A.$$

To demonstrate a typical syntactic use of the rule (Cov), we shall prove the following:

LEMMA 1.1. *CDPDL* $\vdash \langle a \rangle A \rightarrow [a]A$.

Proof.

$$\vdash \langle a \rangle (c \wedge A) \rightarrow \langle a \rangle c \wedge \langle v \rangle (c \wedge A), \text{ by Lemma 3.11}$$

$$\rightarrow [a]c \vee [v](c \rightarrow A), \text{ by (det) and } (\Sigma 2)$$

$$\rightarrow [a]c \wedge [a](c \rightarrow A), \text{ by } (v4)$$

$$\rightarrow [a]A, \text{ by } (K)$$

$$\vdash \neg [a]A \rightarrow \neg \langle a \rangle (A \wedge c), \text{ by contraposition}$$

$$\vdash [(\neg [a]A)?; a; A?] \neg c, \text{ for all } c, \text{ by (?) and (;)}$$

$$\vdash \langle a \rangle A \rightarrow [a]A, \text{ by (Cov) and again (;) and (?).} \quad \blacksquare$$

By the definition of CPDL models, if $\mathscr{M} \models$(det), then R_a will be a (partial) function. Therefore the canonical model will also be a deterministic one. So we immediately obtain

THEOREM 1.2. *Combinatory DPDL is complete with respect to deterministic models.*

Developing the "unwounding" method of (Ben-Ari *et al* 1982), Gargov and Passy (1988) establish the finite model property for CDPDL, proving that any satisfiable formula is satisfiable in a finite model which is, at most, doubly exponential in the size of the formula.

THEOREM 1.3 (Gargov). *CDPDL is finitely complete and decidable.*

Question. Is $\mathscr{F}$CDPDL + (DET), or $\mathscr{F}^*$CDPDL, complete and decidable?

The "atomic" determinism might be combined with the structured constructs IF THEN ELSE and WHILE DO in the so called Structured/ Strict DPDL; see (Halpern and Reif, 1983). Then both completeness for CSDPDL and Ben Ari–Gargov–Halpern–Pnueli–Reif finite completeness results hold again.

1.2. *Limited Finite Models*: $CPDL^{(n)}$

Let diff $(c_1, ..., c_n)$ denote the formula $\bigwedge_{1 \leqslant j<k \leqslant n} \langle c_j \rangle \neg c_k$ which says that $c_1, ..., c_n$ are names of pairwise different states. Note that Segerberg's (1971) scheme

$$(\mathrm{Alt}_n) \quad [\alpha]A_1 \vee [\alpha](A_1 \rightarrow A_2) \vee \cdots \vee [\alpha](A_1 \wedge \cdots \wedge A_n \rightarrow A_{n+1}),$$

which guarantees (for separable models) that each state has less than $n+1$ α-successors, can be replaced by

$$(\textstyle\bigwedge_{j=1}^{n+1} \langle\alpha\rangle c_j) \rightarrow \neg \mathrm{diff}(c_1, ..., c_{n+1}).$$

Let $\mathrm{CPDL}^{(n)} =_{\mathrm{DF}} \mathrm{CPDL} + \{\vdash \neg \mathrm{diff}(c_1, ..., c_{n+1})\}$. The new axiom scheme keeps models' cardinality below $n+1$; we say that these models are *n-models*, and referring to them, define *n-completeness*. (Thus $\mathrm{CPDL}^{(n)}$ is a variant of Segerberg's (1971) S5 Alt_n.) Of course, one has:

THEOREM 1.4. *$CPDL^{(n)}$ is n-complete (and, a fortiori, finitely complete).*

Since we have an upper bound, n, for the refuting model, the non-theorems form a recursive set. So, we get:

THEOREM 1.5. *$CPDL^{(n)}$ is decidable.*

To find the decision complexity (a question posed by the referee), take, with no loss of generality, a closed formula A, fix an n-tuple of pairwise different names $c_1, ..., c_n$ not entering A, and abbreviate

$$A^{(n)} =_{\mathrm{DF}} [v](c_1 \vee \cdots \vee c_n) \rightarrow A.$$

Now, by the n-completeness of $\mathrm{CPDL}^{(n)}$, and by a simple renaming of the model's states, and by the completeness for CPDL, and by the Symmetry lemma I.3.6, one has

$$\mathrm{CPDL}^{(n)} \not\vdash A \qquad \text{iff} \quad \mathrm{CPDL} \not\vdash A^{(n)}.$$

Thus, the deducibility of A over $\mathrm{CPDL}^{(n)}$ is reduced to deducibility of $A^{(n)}$ over CPDL, which is decidable in a deterministic upper bound, exponential in the size of $A^{(n)}$, i.e., exponential in $\mathrm{SIZE}(A) + \mathrm{CONST}(n)$.

We shall continue with this topic, cardinality of models, in Section 3.1 below.

1.3. *The Logic of the Model*

For a model $\mathscr{M}$ let $\mathrm{diag}(\mathscr{M})$ be as in Definition I.2.3. Let $\mathrm{CPDL}^{\mathscr{M}} =_{\mathrm{DF}} \mathrm{CPDL} + \{\mathrm{diag}(\mathscr{M})\}$, i.e., the elements of $\mathrm{diag}(\mathscr{M})$ be added to CPDL as axiom instances (not schemes). We have that $\mathrm{CPDL}^{\mathscr{M}}$ is recursive in $\mathscr{M}$. Let $\vdash_{\mathscr{M}}$ denote provability in $\mathrm{CPDL}^{\mathscr{M}}$.

THEOREM 1.6. *$\vdash_{\mathscr{M}} A$ iff $\mathscr{M} \models A$.*

Proof. The 'only if' part holds by $\mathscr{M}$'s modelhood for $\mathrm{CPDL}^{\mathscr{M}}$. The 'if' part follows from the categoricity of $\mathrm{CPDL}^{\mathscr{M}}$ (Theorem II.4.20), and by the second form of the completeness theorem for $\mathrm{CPDL}^{\mathscr{M}}$. ∎

Note 1.7. A direct inductive proof (not appealing to completeness) exists for the last theorem, from which, on the other hand, the Completeness theorem can be derived as a consequence, as described at the end of Section I.4.

Thus, $\mathrm{CPDL}^{\mathscr{M}} = \{A/\mathscr{M} \models A\}$, and therefore $\mathrm{CPDL}^{\mathscr{M}}$ presents an axiomatization of $\mathscr{M}$'s tautologies, recursive in $\mathscr{M}$. Such a recursive axiomatization is not likely to exist for PDL models (though this question remains open).

However, an analogue of Theorem 1.6 is suggested by Quantified dynamic logic QDL. Below, we dwell on that result, following the exposition of Harel (1984). As tradition goes, reasoning in dynamic logics is usually stratified in three "decreasing" levels of abstractness: propositional, first-order uninterpreted, and first-order interpreted. The third, the most detailed level is claimed to be closest to reasoning about practical programming. At this level, models, called *arithmetic structures*, are rather specified entities whose domain includes a first-order definable copy of the natural numbers ω, and first-order definable functions allowing encoding and decoding of arbitrary finite sequences of the domain into this copy of ω.

The central theorem for Arithmetical completeness of QDL (Harel 1984, Theorem 3.19) states that for any arithmetic structure $\mathscr{A}$, and for any QDL formula A,

$$\mathscr{A} \models A \qquad \text{iff} \quad \text{QDL} + \{F/F \in \text{PC} \,\&\, \mathscr{A} \models F\} \vdash A$$

(we note that QDL includes PC). So Theorem 1.6 turns to be the combinatory version of the Arithmetical completeness theorem, which version holds still on the propositional level. Thus the combinatory facilities suggest a successful (modulo the claims of QDL) abstract reasoning about real programs in concrete models.

2. Σ_1- *and* Π_1-*Definitional Extensions*

Any PC formula $F(\mathbf{c}) = \mathbf{Qe}\, G(\boldsymbol{\alpha}, \mathbf{p}, \mathbf{e}, \mathbf{c})$ of k free variables $c_1, ..., c_k$ performs, in each model $\mathscr{M}$, an operation on $\boldsymbol{\alpha}$ and $\mathbf{p}$, resulting in a k-place predicate of the universe; in this context, we give the following:

Notation 2.1. $R_F^{\mathscr{M}} =_{\text{DF}} \{(\chi(c_1), ..., \chi(c_k))/\mathscr{M} \Vdash F(c_1, ..., c_k)\}$.

When $k = 0$, $R_F^{\mathscr{M}}$ is some (joint) first-order property of the program(s) $\alpha(\cdots)$ mentioned by F, such as transitivity, existence of (ir)reflexive point, equality between programs, universality (equality with ν), etc. When $k = 1$, $R_F^{\mathscr{M}}$ can be thought of as the interpretation of a new formula, and when $k = 2$, as the interpretation of a new program in the language. In particular, all of the operations ;, $\cap$, $\cup$, $\neg$, $^{-1}$, ?, $\subseteq$, $=$ (but not, of course, *) are definable in such a way.

Remark 2.2. (a) We fix $F = \mathbf{Qe}\, G(\boldsymbol{\alpha}, \mathbf{e}, c, d)$; with no loss of generality, we assume the number of F's free variables k to be 2, and skip the unary predicate letters appearing in F. By $\theta_F(\boldsymbol{\alpha})$, or θ, we denote the operation to be axiomatized. The choice of $k = 2$ is made only to force θ have the common arity of a program. So, all considerations from now on apply to the case when each "program" in the language is a predicate of arbitrary finite arity.

(b) In the paper, the axiomatizability-through-expressiveness results spelled out for CPDL will be valid for each definitional extension $\mathscr{E}$ of CPDL liable to the combinatory completeness proof (such as CPDL$^{\cap}$). So, for such $\mathscr{E}$, we shall afford the liberty of identifying $\mathscr{E}$ with CPDL, hence $\mathscr{E}^{\theta}$ with CPDL$^{\theta}$.

To define CPDL$^{\theta}$, we extend the definition of CPDL by the following clauses, respectively:

Syntax of CPDL$^{\theta}$. $\theta(\boldsymbol{\alpha})$ is a program.

Semantics of CPDL$^{\theta}$. $\chi(c)\, R_{\theta}\chi(d)$ iff $\mathscr{M} \Vdash \mathbf{Qe}\, G(\boldsymbol{\alpha}, \mathbf{e}, c, d)$ iff $\mathbf{Qe}(\mathscr{M} \Vdash G(\boldsymbol{\alpha}, \mathbf{e}, c, d)$ (in the last formula $\mathbf{Qe}$ is the informal reading of the quantifiers).

The question now is to find a deductive system generating the tautologies of the language with θ in the models of θ.

2.1. *Axiomatization of Expressible Extensions*

THEOREM 2.3. *Let $A(c, d)$ express $F(c, d)$, cf. Definition* I.1.7. *Then*

$$(Ax_{\theta}) \quad \langle c \rangle \langle \theta(\boldsymbol{\alpha}) \rangle\, d \leftrightarrow A(c, d)$$

yields a complete axiomatization of CPDL$^{\theta}$.

Proof. We simply extend the Truth lemma I.5.1 with the clause

$$\langle c \rangle \langle \theta(\boldsymbol{\alpha}) \rangle\, d \in T \qquad \text{iff} \quad A(c, d) \in T,$$

which is obviously fulfilled, by virtue of (Ax_{θ}).

The definition of the canonical model is extended by

$$R_{\theta(\alpha)} = \{(|c|, |d|) / \langle c \rangle \langle \theta(\boldsymbol{\alpha}) \rangle d \in T\}.$$

So we have

$$\begin{array}{lll} (|c|, \ |d|) \in R_{\theta} & \text{iff} \ \ \langle c \rangle \langle \theta(\boldsymbol{\alpha}) \rangle d \in T & \text{iff} \ \ A(c, d) \in T \\ \text{iff } \mathscr{M} \models A(c, d) & \text{iff } \mathscr{M} \Vdash F(c, d), & \end{array}$$

i.e., $\chi(c)\, R_{\theta}\chi(d)$ iff $\mathscr{M} \Vdash F(c, d)$, and therefore $\mathscr{M}$ is indeed a CPDL$^{\theta}$ model. ∎

Thus we obtain that expressibility implies axiomatizability.

Query 2.4. Does the converse hold (at least in cases when axiomatization is given by a single axiom scheme)?

Conjoining Theorem 2.3 with the Expressiveness theorems I.1.9 and I.1.11, we obtain, respectively, the next two results.

THEOREM 2.5. *Let $G(c, d)$ be an open PC formula, and $\theta = \theta_G$. Then*

$$(Ax_{\theta}) \quad \langle c \rangle \langle \theta(\boldsymbol{\alpha}) \rangle d \leftrightarrow G^{\sim}(c, d)$$

completely axiomatizes CPDL$^{\theta}$.

THEOREM 2.6. *Let $G(c, d)$ be a PC formula without nested quantifiers, and $\theta = \theta_G$. Then*

$$(Ax_\theta) \quad \langle c \rangle \langle \theta(\boldsymbol{\alpha}) \rangle d \leftrightarrow G^{\sim}(c, d)$$

completely axiomatizes $(\mathrm{CPDL} + \{\neg, \bigcap\})^\theta$.

EXAMPLES 2.7. Extensions of CPDL with complement and converse:

$\mathrm{CPDL}^{\neg}$. We extend the definition of CPDL by the additional clauses

Syntax:	$\neg\alpha$ is a program
Semantics:	$R(\neg\alpha) = W^2 \backslash R(\alpha)$
Axiom:	$\langle \neg\alpha \rangle c \leftrightarrow [\alpha] \neg c.$

CPDL^{-1}. We extend the definition of CPDL by the additional clauses

Syntax:	α^{-1} is a program
Semantics:	$R(\alpha^{-1}) = (R(\alpha))^{-1}$
Axiom:	$\langle c \rangle \langle \alpha^{-1} \rangle d \leftrightarrow \langle d \rangle \langle \alpha \rangle c.$

Question 2.8. Are $\mathscr{F}^*\mathrm{CPDL}^{\cap}$ and $\mathscr{F}^*\mathrm{CPDL}^{-1}$ complete? (For $\mathscr{F}^*\mathrm{CPDL}^{\cap}$ the question has been open since (Gargov, 1985)). $\mathscr{F}^*\mathrm{CPDL}^{\neg}$ is not complete, cf. Section 7.

We shall take a small advantage from $\cap$ (recall that $\iota = 1?$ is the identity, and $\alpha^+ =_{\mathrm{DF}} \alpha; \alpha^*$). Let cycle (α) be the predicate for existence of cyclic computation of the program α, formally with the semantics:

$$s \models \mathrm{cycle}(\alpha) \text{ iff } \exists n\, \exists k_{k<n}\, \exists s_1, \ldots, s_n\ (s_1 = s\ \&\ s_k = s_n\ \&\ \forall j_{1 \leqslant j < n}(s_j R_\alpha s_{j+1})).$$

Since $\mathrm{cycle}(\alpha)$ is expressible over $\mathrm{CPDL}^{\cap}$ by the formula $\langle \alpha^* \rangle \langle \alpha^+ \cap \iota \rangle 1$, one obtains, as a side effect,

THEOREM 2.9. *$\mathrm{CPDL}^{\cap}$ + cycle + $\{\vdash \mathrm{cycle}(\alpha) \leftrightarrow \langle \alpha^* \rangle \langle \alpha^+ \cap \iota \rangle 1\}$ is complete.*

2.2. *Axiomatization of the Existential and Universal Operators over CPDL*

In this section, we shall use the power of the quasi-ω-rules to strengthen the result of Theorem 2.3, and to axiomatize CPDL^{θ_F}, for each F which is $\forall^*$- or $\exists^*$-prefixed expressible PC formula. We shall start with two examples: inclusion and equality of programs.

CPDL$^{\subseteq}$. We extend the definition of CPDL by the additional clauses

Syntax: $\alpha \subseteq \beta$ is a formula.

Semantics: $s \models (\alpha \subseteq \beta)$ iff $\forall t\ (sR_\alpha t$ implies $sR_\beta t)$

Axiomatics: $(Ax_{\subseteq})\ \alpha \subseteq \beta \rightarrow (\langle \alpha \rangle c \rightarrow \langle \beta \rangle c)$

(Incl) If $\vdash [\gamma](\langle \alpha \rangle c \rightarrow \langle \beta \rangle c)$, for all c, then $\vdash [\gamma]\alpha \subseteq \beta$.

We should now require that theories be closed under the new quasi-ω-rule (Incl), accordingly re-prove the Separation lemma, and add a new clause to the Truth lemma:

$$\textcircled{c}\ \alpha \subseteq \beta \in T \quad \text{iff} \quad \forall d(\textcircled{c}\ \langle \alpha \rangle d \in T \quad \text{only if} \quad \textcircled{c}\ \langle \beta \rangle d \in T).$$

Finally, we obtain:

THEOREM 2.10. *CPDL$^{\subseteq}$ is complete.*

Note 2.11. Since $\alpha \cap \beta = \neg(\neg\alpha \cup \neg\beta)$, and $\alpha \subseteq \beta$ iff $\alpha \cap \neg\beta = \varnothing$, $\alpha \subseteq \beta$ turns to be expressible over CPDL$^{\neg}$, and therefore, by Theorems 2.3 and 2.5 (cf. Remark 2.2.b above), axiomatization of $\subseteq$ is obtained, over CPDL$^{\neg}$, by a single axiom: $\alpha \subseteq \beta \leftrightarrow [\alpha \cap \neg\beta]0$.

Under self-explanatory definitions for equality of programs, $=$, one has:

THEOREM 2.12. *CPDL$^{=}$ is complete.*

Thus we reached axiomatizations for a large part of relational-algebraic operations: ;, $\cup$, $\cap$, $\neg$, $^{-1}$, $\subseteq$, $=$, ?, *, ν, ι. However, the expressive limits of the model language encompassing these operations are still far from covering all formulae with two nested quantifiers, let alone all $\forall$*- and $\exists$*-cases, as we know by Proposition I.1.12. So the next theorems will give many more axiomatizations than those guaranteed by expressiveness.

Let F be any PC formula of the type $F(c, d) = \exists \mathbf{e}\, G(\boldsymbol{\alpha}, \mathbf{e}, c, d)$, where G is an arbitrary *expressible* formula and let $G^{\sim}$ be a CPDL formula expressing G. In the case of open G, i.e., existential F, we have $G^{\sim} \in$ CPDL, cf. Section I.1, and if G has no nested quantifiers, we have some $G^{\sim} \in$ CPDL $+ \{\cap, \neg\}$.

Now we are ready to start with the axiomatization of CPDL$^{\theta}$, for $\theta = \theta_F(\boldsymbol{\alpha})$. We have the specific for CPDL$^{\theta}$ clauses:

Syntax: $\theta(\boldsymbol{\alpha})$ is a program

Semantics: $\chi(c)\, R_\theta \chi(d)$ iff $\exists \mathbf{e}(\mathcal{M} \Vdash G(\boldsymbol{\alpha}, \mathbf{e}, c, d))$

and we add, on the deductive side,

Axiomatics: (Ax_θ) $G^\sim(\mathbf{a}, \mathbf{e}, c, d) \to \text{ⓒ}\langle\theta(\mathbf{a})\rangle d$

(R_θ) If $\vdash [\gamma]\neg G^\sim(\mathbf{a}, \mathbf{e}, c, d)$,
for all $\mathbf{e} \in \Sigma \times \cdots \times \Sigma$,
then $\vdash [\gamma](c \to [\theta(\mathbf{a})]\neg d)$.

Exercise 2.13. Prove the soundness of CPDL$^\theta$.

Note 2.14. The rule (R_θ) is a quasi-ω-rule for CPDL$^\theta$.

We define the CPDL$^\theta$ deductive sets to be closed also under (R_θ), reprove the Separation lemma, and extend the Truth lemma by:

TRUTH LEMMA 2.15. ⓒ$\langle\theta(\mathbf{a})\rangle d \in T$ *iff* $\exists\mathbf{e}\ (G^\sim(\mathbf{a}, \mathbf{e}, c, d) \in T)$.

Proof. (if) Follows immediately by (Ax_θ).

(only if) Let ⓒ$\langle\theta(\mathbf{a})\rangle d \in T$. Suppose that $\forall\mathbf{e}(G^\sim(\mathbf{a}, \mathbf{e}, c, d) \notin T)$. Hence, $\forall\mathbf{e}\ (\neg G^\sim(\mathbf{a}, \mathbf{e}, c, d) \in T)$. So, by (R_θ) with $\gamma = v$, ⊡$(c \to [\theta(\mathbf{a})]\neg d) \in T$, and so ⓒ$[\theta(\mathbf{a})]\neg d \in T$, which contradicts ⓒ$\langle\theta(\mathbf{a})\rangle d \in T$. ∎

LEMMA 2.16. *Let T be a maximal theory, and $\mathcal{M}$ the canonical model of T. Then $\mathcal{M} \Vdash G(\mathbf{a}, \mathbf{e}, c, d)$ iff $G^\sim(\mathbf{a}, \mathbf{e}, c, d) \in T$.*

Proof. Since $G^\sim$ expresses G, i.e., $\mathcal{M} \Vdash G$ iff $\mathcal{M} \models G^\sim$, by the definition of $\mathcal{M}$, $\mathcal{M} \Vdash G$ iff $G^\sim \in T$. ∎

COROLLARY 2.17. $\exists\mathbf{e}\ (G^\sim(\mathbf{a}, \mathbf{e}, c, d) \in T)$ *iff* $\exists\mathbf{e}\ (\mathcal{M} \Vdash G(\mathbf{a}, \mathbf{e}, c, d))$ *iff* $(\mathcal{M} \Vdash \exists\mathbf{e}\, G(\mathbf{a}, \mathbf{e}, c, d))$.

COROLLARY 2.18. ⓒ$\langle\theta(\mathbf{a})\rangle d \in T$ *iff* $\mathcal{M} \Vdash F(c, d)$.

Proof. By the Truth lemma 2.15, and Corollary 2.17. ∎

Corollary 2.18 guarantees that the canonical model is a model for CPDL$^\theta$. Whence:

THEOREM 2.19.a. CPDL$^\theta$ *is complete.*

Now, let $\theta = \theta_F$ be defined through a universalization F of an expressible G: $F = \forall\mathbf{e}\, G$. Since θ_F is expressible over CPDL$^{\theta_{\neg F}}$, we can, by Theorem 2.3, axiomatize θ_F over CPDL$^{\theta_{\neg F}}$ postulating $\vdash \langle\theta_F\rangle d \leftrightarrow [\theta_{\neg F}]\neg d$. For independent axiomatization of universal θ we may give, analogically to the previous theorem, the obvious axiom and quasi-ω-rule, and obtain

THEOREM 2.19.b. *$CPDL^{\theta}$ is complete.*

To effectively use the full power of Theorems 2.3 and 2.19, one needs an explicit description of the sets of expressible PC formulae and so we once again stress on Problems I.1.13.

Question 2.20. Theorems 2.19 give only a sufficient condition for axiomatizability of (Ax_{θ}) & (R_{θ})-type. We should mention a development of the axiomatizability-through-expressiveness idea, which would generalize "expressiveness" up to "Σ- and Δ-expressiveness" (i.e., expressibility of first- and second-order conditions by infinite disjunctions and conjunctions of CPDL formulae, instead of expressibility by a single CPDL formula), and extract further axiomatic benefits from those kinds of expressiveness. Examples of that kind are given in the next section.

3. *Infinities in Computations*

One can hardly speak of computation without touching infinity, and to approach infinity, human beings have devised iteration. Dynamic logics, too, exemplify that rule: the principal super-modal feature of PDL is the iteration *: $[\alpha^*]A \leftrightarrow \forall n([\alpha^n]A)$. Likewise, in Algorithmic logic starring is another iterative construct, the iteration quantifier: $(\Pi\alpha)A \leftrightarrow \forall n(\langle\alpha^n\rangle A)$. Instantiating 1 for A in the last formula, we obtain the predicate $(\Pi\alpha)1$ or $\Pi\alpha$ for short, for existence of arbitrary long paths of α-successors. Besides $\Pi\alpha$, there are two more notions natural in this context.

One is the predicate, denote it by ∇, for existence of arbitrarily wide (hence infinitary) fans of α-successors: $\nabla\alpha \leftrightarrow \forall n(\langle\alpha\rangle_n 1)$, where $x \models \langle\alpha\rangle_n 1$ holds when x has at least n different (immediate) α-successors. (These $\langle\alpha\rangle_n$ have been systematically studied—under the name of *graded modalities*—by the Roma group, see (Fattorosi-Barnaba and Carro, 1985).)

The other is the predicate Δ, for existence of infinite path of α-successors, which in rough notation might be expressed by $\Delta\alpha \leftrightarrow \langle\alpha^{\omega}\rangle 1$. These three predicates Π, Δ, ∇ and the operation * are naturally encapsulated in the famous Brouwer–König lemma: $\Pi\alpha \rightarrow \Delta\alpha \vee \langle\alpha^*\rangle\nabla\alpha$. In this section we apply the combinatory machinery to that software of infinity.

3.1. *Bounds on Non-determinism and Cardinality*

We shall pay attention here to the predicate $\langle\alpha\rangle_n A$ (for $2 \leqslant n \leqslant \omega$)—a generalization of $\langle\alpha\rangle A$—which is true of a state s, if s has at least n different α-successors satisfying A. (Appealing to the tautology $\langle\alpha\rangle_n A \leftrightarrow \langle\alpha; A?\rangle_n 1$, we shall focus on the case $\langle\alpha\rangle_n 1$.) Consequently, the negation of $\langle\alpha\rangle_n 1$, denoted by $[\alpha]_n 0$, is satisfied in s, if s has less than n α-successors. In the case $n = \omega$, denoting $\langle\alpha\rangle_{\omega} 1 =_{\mathrm{DF}} \nabla\alpha$, we get a control over the finite/infinite non-determinism of α, through $\neg\nabla\alpha/\nabla\alpha$. Instantiating v for α, we reach a single-formula-control over the cardinality of the model: ∇v, $\neg\nabla v$, $[v]_{n+1}0$ guarantee, respectively, that the model is infinite, finite, or n-model. We recall the formula $\mathrm{diff}(c_1 \cdots c_n)$ defined in Section 1.2.

For each n, $1 \leqslant n < \omega$, we define CPDL$^{\diamond_n}$ as an extension of CPDL, by the syntactic clause "$\langle\alpha\rangle_n A$ is a formula," with semantics as defined above, and axiomatics

$$(\mathrm{Ax}_{\diamond_n}1) \vdash \left(\mathrm{diff}(c_1, ..., c_n) \wedge \bigwedge_{j=1}^{n} \langle\alpha\rangle c_j\right) \rightarrow \langle\alpha\rangle_n 1$$

$$(\mathrm{Ax}_{\diamond_n}2) \vdash [\alpha](c_1 \vee \cdots \vee c_{n-1}) \rightarrow [\alpha]_n 0.$$

Theorem 3.1. *CPDL$^{\diamond_n}$ is complete.*

THEOREM 3.2. *CPDL$^{\diamond_n}$ + $\{\vdash [v]_n 0\}$ is n-complete, and, therefore decidable.*

For $n = \omega$, we have $\nabla\alpha \leftrightarrow \forall n\ (\langle\alpha\rangle_n 1)$, and so we introduce CPDL$^{\nabla}$ as a definitional extension of the union of all CPDL$^{\diamond_n}$, $n < \omega$, adding the proper clauses:

Syntax of CPDL$^{\nabla}$: $\nabla\alpha$ is a formula.

(Ax^n_{∇}) $\nabla\alpha \rightarrow \langle\alpha\rangle_n 1$, for each $n < \omega$;

(R_{∇}) If $\vdash [\gamma]\langle\alpha\rangle_n 1$, for all $n < \omega$, then $\vdash [\gamma]\nabla\alpha$.

THEOREM 3.3. *CPDL$^{\nabla}$ is complete.*

Let CPDL$^{\mathrm{fin}} =_{\mathrm{DF}}$ CPDL$^{\nabla}$ + $\{\vdash \neg\nabla v\}$, and CPDL$^{\mathrm{inf}} =_{\mathrm{DF}}$ CPDL$^{\nabla}$ + $\{\vdash \nabla v\}$. Then, $\mathscr{M}$ is a model for CPDL$^{\mathrm{fin(inf)}}$ iff $\mathscr{M}$ is a finite (infinite) CPDL$^{\nabla}$-model.

THEOREM 3.4. (a) *CPDL$^{\mathrm{fin}}$ is complete.*

(b) *CPDL$^{\mathrm{inf}}$ is complete.*

As far as decidability is concerned, we can only say that the formula ∇v disproves the fmp for CPDL$^{\nabla}$, whereas, by definition, CPDL$^{\mathrm{fin}}$ has the fmp, and CPDL$^{\mathrm{inf}}$ lacks this nice property.

Questions. Are CPDL$^{\diamond_n}$, CPDL$^{\nabla}$, CPDL$^{\mathrm{fin}}$, CPDL$^{\mathrm{inf}}$ decidable?

3.2. *The Polish Iteration Quantifier*

We add, over CPDL, the operation Π on programs, defined by $\Pi\alpha = \bigcap_{n<\omega} \alpha^n$. This equivalence immediately suggests the axiomatization

(Ax^n_{Π}) $(\Pi\alpha)A \rightarrow \langle\alpha^n\rangle A$ (for all $n < \omega$); and

(R_{Π}) If $\vdash [\gamma]\langle\alpha^n\rangle A$, for all $n < \omega$, then $\vdash [\gamma](\Pi\alpha)A$.

As a direct application of the completeness techniques exploited above, we immediately obtain:

THEOREM 3.5. *$CPDL^{\Pi}$ is complete.*

3.3. *Repeating*

The *repeating*-α predicate, $\Delta\alpha$, defined by

$$s \models \Delta\alpha \qquad \text{iff} \quad \exists s_0, s_1, \ldots\ (s_0 = s \ \&\ \forall k_{k<\omega}(s_k R_\alpha s_{k+1}))$$

has become renowned in this decade among dynamic and modal logicians because of its insusceptibility to axiomatic treatment. A hypothesis of Streett (1982) suggests that the two axioms

$$(\mathrm{Ax}_\Delta^1) \quad \Delta\alpha \rightarrow \langle\alpha\rangle \Delta\alpha,$$
$$(\mathrm{Ax}_\Delta^2) \quad A \wedge [\alpha^*](A \rightarrow \langle\alpha\rangle A) \rightarrow \Delta\alpha$$

yield a complete axiomatization of Δ, over PDL. (A recent, still unpublished, paper of Sakalauskaite and Valiev (1989) claims to have proved Streett's conjecture.) The combinatory approach makes an advance in this conjecture, proving it in a combinatory setting.

Let $\mathrm{CPDL}^\Delta =_{\mathrm{DF}} \mathrm{CPDL} + \{\Delta\} + (\mathrm{Ax}_\Delta^1) + (\mathrm{Ax}_\Delta^2)$. We have:

THEOREM 3.6 (Gargov and Passy, 1988). *$CPDL^\Delta$ is complete, decidable, and has the fmp.*

Remark 3.7. The proof of this last theorem considerably deviates from the other completeness proofs presented thus far: an intermediate step of the proof appeals to the infinitary rule

(R_Δ) If $\vdash [\gamma] \Diamond_{c_k} \langle\alpha\rangle c_{k+1}$, for some infinite string of names $c_0, c_1, \ldots$, then $\vdash [\gamma] \Diamond_{c_0} \Delta\alpha$,

which in fact has the shape of a β-rule, and not of an ω-one.

Thus (R_Δ) requires an essential modification of the notion of theory, and the proof of the Separation lemma than relies on the Brouwer–König lemma.

Open Question 3.8. Had we devised some general proof-theoretic method to establish conservativeness of a combinatory system over its non-combinatory fragment, we would have immediately obtained proof of Streett's conjecture for PDL^Δ. Moreover, such a method would automatically induce all completeness results on the non-combinatory counterparts of the systems we treat.

Harel (1984) suggests an infinitary axiomatization of Δ over QDL, by the axiom $\Delta\alpha \leftrightarrow \langle\alpha\rangle \Delta\alpha$, and the ω-rule if $\vdash A \to \langle\alpha^n\rangle 1$, for $n < \omega$, then $\vdash A \to \Delta\alpha$. It is interesting to observe that this rule is a version of (R_Π) from the previous sub-section, which yields axiomatiation of Π.

3.4. *The Formal Brouwer–König Lemma*

THEOREM 3.9. $CPDL^{\Pi,\nabla,\Delta} \vdash \Pi\alpha \to (\Delta\alpha \vee \langle\alpha^*\rangle \nabla\alpha)$.

Proof. By the completeness theorem for CPDL$^{\Pi,\nabla,\Delta}$. ∎

4. *Two Extensions with Choice Function*

By (uniform) *selector* on a binary relation, we mean a function which is included in the relation and has the same domain. In recursion theory, the uniformization theorem (known also as the single-valuedness theorem, or the selector theorem) says that for each recursive (resp., Π^1_1) relation there is a recursive (resp., Π^1_1) selector, cf. (Rogers, 1967; Shoenfield, 1967). This elegant theorem plays an important role in studying generalized computability. In view of the traditional conceptual impact of theory of computability on logics of programs, the selector seems a natural innovation in the ambience of PDL. In this section, we apply the combinatory method to axiomatization of the selector on programs. Next, we apply the selector to an axiomatic approach to the well-ordering relation.

4.1. *Uniform Selector on Programs: CPDL$^\rho$*

We define CPDL$^\rho$ as an extension of CPDL$^\subseteq$ (see Section 2.2) adding—for a fresh letter ρ—the specific clauses:

Syntax of CPDL$^\rho$: $\rho\alpha$ is a program.

Semantics: $\mathcal{M} = (M, R, \chi, V, r)$ is a CPDL$^\rho$-model, if (M, R, χ, V) is a CPDL$^\subseteq$-model, and $r: M \times (2^M \setminus \{\varnothing\}) \to M$ is a choice function,

i.e., $r(s, X) \in X$, for $\varnothing \neq X \subseteq M$; and

$$sR_{\rho\alpha}t \quad \text{iff} \quad r(s, R_\alpha(s)) = t.$$

So, we have defined $R_{\rho\alpha}$ to be a function included in R_α, with the same domain.

Axioms for ρ:

$(\rho 1)$ $\langle\alpha\rangle 1 \to \langle\rho\alpha\rangle 1$

$(\rho 2)$ $\rho\alpha \subseteq \alpha$

$(\rho 3)$ $\langle\rho\alpha\rangle c \to [\rho\alpha] c$

$(\rho 4)$ $\alpha \subseteq \beta \,\&\, \beta \subseteq \alpha \to \rho\alpha \subseteq \rho\beta \,\&\, \rho\beta \subseteq \rho\alpha$

We extend the Truth lemma accordingly. In the canonical model $\mathcal{M}$, we define the choice function $r(|c|, X)$ for each $|c| \in M$ and X, $\varnothing \neq X \subseteq M$, as follows: if $\exists \alpha\ (X = R_\alpha(|c|))$, then, by the Truth lemma, there is some d with $\textcircled{c}\langle\rho\alpha\rangle d \in T$, and we postulate $r(|c|, X) = |d|$; if $\neg\exists\alpha\ (X = R_\alpha(|c|))$, then we choose $r(|c|, X) \in X$ arbitrarily.

THEOREM 4.1. *$CPDL^\rho$ is complete.*

4.2. *The Logic of Well-ordering*: $CPDL^{\leqslant}$

For a fresh letter λ, we add over $CPDL^\rho$ the new clauses proper for λ:

Syntax of $CPDL^{\leqslant}$: λ is a program.

Semantics of $CPDL^{\leqslant}$: a model for $CPDL^{\leqslant}$ is any $CPDL^\rho$-model, in which the following conditions are met ($s \leqslant t$ denotes $sR_\lambda t$):

(i) $\leqslant$ is an antisymmetric and transitive relation;

if $\varnothing \neq X \subseteq M$, and $\exists\alpha\,\exists t\ (X = R_\alpha(t))$, then

(ii) X has a least element with respect to $\leqslant$, and

(iii) $r(s, X)$ chooses the least element of X (for any s).

Axioms for λ:

(λ1) $\textcircled{c}\langle\lambda\rangle d \wedge \textcircled{d}\langle\lambda\rangle c \rightarrow \textcircled{c}\, d$

(λ2) $\textcircled{c}\langle\lambda\rangle e \wedge \textcircled{e}\langle\lambda\rangle d \rightarrow \textcircled{c}\langle\lambda\rangle d$

(λ3) $\langle\alpha\rangle c \wedge \langle\rho\alpha\rangle d \rightarrow \textcircled{d}\langle\lambda\rangle c$.

THEOREM 4.2. *$CPDL^{\leqslant}$ is complete.*

5. *Polyadic Extensions*

The possible worlds semantics for modal logic is based on the accessibility relation R, which can be thought of as a monadic (and multi-valued) function saying, for any worlds s and t, whether t is a possible development of s. This formalism is a special case of various more general approaches utilizing polyadic (many-argument) accessibility functions or, moreover, acessibility relations, bounding collections, ordered or not, of possible worlds. We shall not undertake here the development of these approaches in the nominated environment, but will only make two steps in that challenging direction.

A common stream of both what has been done in previous sections and what might be sought in the generalization in question may be layed down in a view expressed by Došen (1986) who calls modal logic a "*general* theory of unary propositional operators" and asks "Is modal logic able to deal with arbitrary unary propositional operators?" Relating that question to combinatory modal logic, in the light of the axiomatizability results for universal- and existential- and for arithmetical-operators (Theorems 2.19 and III.2.3), the answer tends to be positive. The same author, in the same paper, appeals for the creation of a logical theory able to claim the honourable title of "general theory of *binary* propositional connectives." Furthermore, he dreams that this "theory should cover not only two-valued, or many-valued, or intuitionistic, or relevant connectives, but any connectives we might wish to consider."

"Binarity" of a certain operator θ might be understood as "diadicity," i.e. $\theta = \theta(A, B)$, as well as "2-dimensionality," i.e., as a request for the evaluation (of formulae containing θ) to be made in two points of the universe simultaneously. (The k-arity has, accordingly, the same meanings.) Regarding both possibilities, the combinatory approach makes no difference between cases $k=2$ and $k \neq 2$ (cf. Remark 2.2), and so combinatory modal logic runs as a plausible candidate for Došen's honourable title and a materialization of his dream. To illustrate this, we shall give two applications of the combinatory method: one treating polyadicity and another treating multi-dimensionality.

5.1. *Pairing of the States*: *CPDL^J*

We again make the stipulation that CPDL^J is an extension of CPDL, mentioning only the new clauses:

Syntax of CPDL^J. If A, B are formulae, then $J(A, B)$ is a formula.

Semantics. $\mathcal{M} = (M, R, \chi, V, j)$ is a CPDL^J-model, if (M, R, χ, V) is a CPDL-model, and $j \subseteq M \times M \times M$ is a ternary relation (with the intended meaning of multi-valued coding function j: $M^2 \to M$). The satisfiability relation is extended by

$$s \models J(A, B) \quad \text{iff} \quad \exists u \, \exists v (u \models A \,\&\, v \models B \,\&\, (u, v, s) \in j),$$

i.e., $V(J(A, B))$ consists of the j-codes of the pairs from $V(A) \times V(B)$.

To bring the relation j closer to the usual notion of coding functions, one may optionally add some of the following constraints:

1. j is a function (from M^2 into M)
2. j is totally defined, i.e., $\mathrm{dom}(j) = M^2$
3. j is injective, i.e., $j(s, t) = j(u, v)$ implies $s = u$ & $t = v$
4. j is surjective, i.e., range $(j) = M$.

Axiomatics.

(Ax_J) $\langle c \rangle A \wedge \langle d \rangle B \wedge J(c, d) \rightarrow J(A, B)$

(R_J) If $\vdash [\gamma] \neg(\langle c \rangle A \wedge \langle d \rangle B \wedge J(c, d))$, for all c, d, then $\vdash [\gamma] \neg J(A, B)$.

Having adopted any of constraints 1–4, one should add, respectively, the axioms:

1. $\langle e \rangle J(c, d) \wedge \langle e_1 \rangle J(c, d) \rightarrow \langle e \rangle e_1$
2. $\Diamond J(c, d)$
3. $J(c, d) \wedge J(c_1, d_1) \rightarrow \langle c \rangle c_1 \wedge \langle d \rangle d_1$
4. $J(1, 1)$.

For the Truth lemma, we add

$$\langle c \rangle J(A, B) \in T \quad \text{iff} \quad \exists d \exists e(\langle d \rangle A \wedge \langle e \rangle B \wedge \langle c \rangle J(d, e) \in T).$$

In the canonical model, we define

$$(\chi(c), \chi(d), \chi(e)) \in j \quad \text{iff} \quad \langle e \rangle J(c, d) \in T.$$

THEOREM 5.1. *$CPDL^J$ is complete.*

We shall not discuss the left and right decoding functions for j: $l(j(s, t)) = s$ and $r(j(s, t)) = t$, whose axiomatic treatment in this framework is also a routine matter.

5.2. *Many-Dimensional Dynamic Logic*

Segerberg (1973) considers "two-dimensional" modal logic as a natural tool for the investigation of several instances of "two-dimensional" modalities, namely modal operators requiring evaluation in pairs of states. A motivation for doing so comes from temporal logic, from properties possessible by pairs of moments of time. For example, if (s, t) is the period of time locked between moments of time s and t, a sample two-dimensional property P would be

$$P(s, t) \quad \text{if} \quad (s, t) \text{ is shorter than } (s_0, t_0), \text{ for some fixed } s_0, t_0.$$

Paying debts to the computational background of the whole enterprise, one might call the argument (adduced by Harel (1983) in motivating dominoes) that 2-dimensional modal logic "can be regarded as an appealing abstraction capturing the two-dimensional time/space character of computation, but one which is devoid of the details of particular computing machines." Of course, some other magical properties of the holy number 2 might be spelled as well.

Segerberg's paper gives a formal approach to two-dimensional phenomena, proving completeness and decidability. The completeness proof is rather intricate and the author concludes the paper with an appeal for a much shorter proof. This question was later put by D. Vakarelov in a more general setting: to find an axiomatization for the n-dimensional case (Segerberg's method already fails for $n = 3$, cf. (Vakarelov, 1985–1988), and, moreover, to do this over PDL, where n-argument programs prove quite natural objects. We shall outline the solution of A. Petkov (A. Petkov, 1987) axiomatizing the n-dimensional PDL (for each fixed $n < \omega$) in the typical combinatory spirit. Up to notation and minor simplifications, the rest of this section is a sketch of this solution; we exemplify the method for $n = 2$, which case makes no difference from any other Cartesian degree $n < \omega$.

Square CPDL, SqCPDL, will be the system dealing with the 2-dimensional case.

The syntax of SqCPDL is defined as an extension of the basic CPDL language with two new modalities (programs) $\langle x \rangle$ and $\langle y \rangle$ (say x, $y \in \Pi_0$), and one new propositional constant E (say $E \in \Phi_0$).

Semantics of SqCPDL. A *square model* is any quintuple $\mathcal{M} = (M, U, R, \chi, V)$ such that (M, R, χ, V) is a CPDL model, and

$$M = U^2;$$

E is interpreted as the diagonal of M, i.e., $V(E) = \{(s, s)/s \in U\}$; x, y are interpreted as decoding functions in the sense that $R_x((s, t)) = (s, s)$ and $R_y((s, t)) = (t, t)$, for each s, $t \in U$.

The Square Axioms, where z runs over $\{x, y\}$, are:

(A1.1) $\langle z \rangle c \rightarrow [z] c$

(A1.2) $\langle z \rangle E$

(A2) $\langle c \rangle E \wedge \langle d \rangle E \rightarrow \odot(\langle x \rangle c \wedge \langle y \rangle d)$

(A3) $\langle c \rangle(\langle x \rangle e_1 \wedge \langle y \rangle e_2) \wedge \langle d \rangle(\langle x \rangle e_1 \wedge \langle y \rangle e_2) \rightarrow \langle c \rangle d$

(A4) $c \wedge E \rightarrow \langle z \rangle c$

(A5) $\langle x \rangle d \wedge \langle y \rangle d \rightarrow E.$

THEOREM 5.2 [Petkov, 1987]. *SqCPDL is complete.*

The Square CPDL and its intersected version, SqCPDL$^\cap$, turn out to provide a suitable environment for a good deal of specific two-dimensional operators. For example, the following are uniformly axiomatizable (where $R(u, v)$ is $R((u, v))$):

$$R_1(u, v) = \{(v, u)\}$$

$$R_2(u, v) = \{(u, w)/w \in U\}$$

$$R_3(u, v) = \{(w, v)/w \in U\}$$

$$R_4(u, v) = \{(v, w)/w \in U\}$$

$$R_5(u, v) = \{(w, u)/w \in U\}.$$

Note. The modalities $[R_1]$, $[R_2]$, $[R_3]$, as well as $[x]$, $[y]$, $[v]$, are taken as a necessary part of the ground language in Segerberg's approach.

Left open is the question of a general theorem, in the spirit of Theorems 2.19, describing some nice (possibly the whole) class of axiomatizable diadic relations over the Square models.

6. *Undecidability and Finitary Incompleteness*

The infinitary inference rules, as opposed to r.e. axiomatics, guarantee as low as Π^1_1-completeness, rather than Σ^0_1, for an upper bound on the hardness of provability problem.

DEFINITION 6.1. For an axiomatic system $\mathscr{E}$, we say that $\mathscr{E}$ is *very hard*, if deciding provability in $\mathscr{E}$ is Π^1_1-complete. On the semantical side, we say that $\mathscr{E}$ is *highly undecidable*, if the deciding satisfiability in $\mathscr{E}$ is Σ^1_1-complete.

Provided $\mathscr{E}$ is Kripke-complete, $\mathscr{E}$ is very hard iff $\mathscr{E}$ is highly undecidable. We shall apply Harel's (1983) domino-method to some of the systems from this chapter to show them highly undecidable, whence very hard. In fact, we shall refer to the high undecidability of the PDL-based counterparts, which negative result—being automatically induced on extensions—will apply to the combinatory extensions of those systems. Another inheritable negative property is the lack of fmp, and first we state this for several combinatory cases.

Take, cf. Section 2.1, $\text{cycle}(a) =_{\text{DF}} \langle a^* \rangle \langle a^+ \cap \iota \rangle 1$, and, cf. (Harel, 1984),

$$A_\cap =_{\text{DF}} [a^*]\langle a \rangle 1 \rightarrow \text{cycle}(a),$$

which formula is obviously true in each finite $PDL^{\cap}$ (or $CPDL^{\cap}$) model and which obviously fails in (each point of) the infinite model $\mathcal{M}_0 = (N, R, V)$, where $N = \{0, 1, 2, ...\}$, $R_a = \{(k, k+1)/k \in N\}$, V is arbitrary. So the formula $A_{\cap}$, i.e., the presence of the pair "$\cap$, *" in the language, excludes the fmp. Hence, the pair "$\neg$, *" would do as much. So would do the pairs "$\subseteq$, *" and "$=$, *", as could be seen from the same model $\mathcal{M}_0$, and, respectively, the formulae

$$A_{\subset} =_{DF} [a^*]\langle a \rangle 1 \to \langle a^* \rangle(\iota \subseteq a^+), \qquad \text{and}$$
$$A_{=} =_{DF} [a^*]\langle a \rangle 1 \to \langle a^* \rangle(a^+ = a^*).$$

Summarizing these, and the observations from Section 3.1, we can formulate

THEOREM 6.2. *$CPDL^{\#}$, where $\# \in \{\cap, \neg, \subseteq, =, \rho, \leqslant, \nabla, \inf\}$, lacks the fmp.*

In the next theorem (and later, in the next Chapter) we shall refer to a result of Harel and Vardi, precisely to its domino setting, see (Harel, 1983, Theorem 4.9), stating the high undecidability of Deterministic $PDL^{\cap}$. To establish this, determinism is exploited only to the extent it guarantees the existence of two deterministic program variables a and b, and intersection is needed to guarantee that composition somewhat commutes over these a and b: $\langle ab \cap ba \rangle 1$. Since languages with $\Diamond_2$ are able to guarantee determinism: $\boxdot(\neg\langle a \rangle_2 1 \wedge \neg\langle b \rangle_2 1)$, and, by definition, ρa and ρb are deterministic, we have, as a direct corollary to (Harel 1983, Theorem 4.9), the following:

THEOREM 6.3. *$CDPDL^{\cap}$, $CSDPDL^{\cap}$, $CPDL^{\Diamond_2, \cap}$, $CPDL^{\rho, \cap}$, $CPDL^{\leqslant, \cap}$ are highly undecidable, hence very hard.*

As an application of the domino method we also have the next two theorems, the third being an immediate consequence of them.

THEOREM 6.4 [Gargov, 1984–1986]. *$PDL^{\subset, a, -1}$ is highly undecidable (here a^{-1} is the converse of a single atomic program).*

THEOREM 6.5. [Gargov, 1984–1986]. *$PDL^{\cap, \neg\iota}$ is highly undecidable (here $\neg\iota$ is the complement of the identity program ι).*

THEOREM 6.6. *$CPDL^{\subset, a: -1}$, $CPDL^{\subset, -1}$, $CPDL^{\cap, \neg\iota}$, $CPDL^{\neg}$ are highly undecidable, hence very hard.*

We mentioned here some "negative" decidability results which, in addition to the "positive" ones obtained above (decidability of CPDL, $\mathrm{CPDL}^{(n)}$, $\mathrm{CPDL}^{\diamond_k} + \{\vdash \boxdot_k 0\}$, CPDL^{Δ}, and fmp for those and $\mathrm{CPDL}^{\mathrm{fin}}$) exhaust our present knowledge on that matter. To round up the discussion, we shall raise some questions that remained open.

Questions 6.7. a. Fmp for CPDL^{-1}, $\mathrm{CPDL}^{\diamond_k}$, for $k > 2$, CPDL^{Π}, CPDL^{J}, SqCPDL.

b. Decidability for $\mathrm{CPDL}^{\cap}$, CPDL^{-1}, $\mathrm{CPDL}^{\subset}$, $\mathrm{CPDL}^{=}$, $\mathrm{CPDL}^{\diamond_k}$, CPDL^{∇}, $\mathrm{CPDL}^{\mathrm{fin}}$, $\mathrm{CPDL}^{\mathrm{inf}}$, CPDL^{Π}, CPDL^{ρ}, $\mathrm{CPDL}^{\leqslant}$, CPDL^{J}, SqCPDL. We should stress on the intersection, $\mathrm{CPDL}^{\cap}$, which, in the light of the decidability of $\mathrm{PDL}^{\cap}$, cf. (Danecki, 1985), we expect to be decidable. However, we are not aware of a general method reducing the decidability of a nominated modal system to decidability of the non-combinatory counterpart.

c. Completeness and decidability of the $\mathscr{F}$- and $\mathscr{F}$*-, cf. I.7.10, versions of the systems from this chapter. Again starring are the intersection "Is $\mathscr{F}$*$\mathrm{CPDL}^{\cap}$ complete?" and determinism, "Are $\mathscr{F}$CDPDL + (DET) and $\mathscr{F}$*CDPDL complete and decidable?"

Neglecting the β-rule (R_{Δ}) from Remark 3.7, the remaining of the *long rules* are ω-ones, and depending on the "type of infinity" they induce, are either (Cov)-like, with quantifier $\forall c_{c \in \Sigma}$, or (Ind)-like, with quantifier $\forall n_{n < \omega}$. In most cases, the (Cov)-type infinity proves to be a quasi-infinity. As it was mentioned in I.3.9, for a symmetric system $\mathscr{E}$ (and all considered thus far, save CPDL^{M}, are such) we have it that $\mathscr{E} = {}^*\mathscr{E}$, i.e.,

PROPOSITION 6.8. *Provided $\mathscr{E}$ is symmetric, $\mathscr{E}$ is complete iff ${}^*\mathscr{E}$ is complete.*

The attempts to finitize (Ind)-type infinity, however, would in many cases fail. Extending convention I.7.10, we make the following

Convention 6.9. Given a system $\mathscr{E}$ with an ω-rule

$$\text{(R) If } \vdash A\{n\}, \text{ for all } n < \omega, \text{ then } \vdash B,$$

by $\mathscr{F}\mathscr{E}$ we shall denote any version of $\mathscr{E}$, in which each such (R) is replaced by some finitary (admissible) rule, or by an r.e. set of axioms (provable in $\mathscr{E}$).

Now, $\mathscr{F}{}^*\mathscr{E}$ will no more have ω-rules, and will have an r.e. set of theorems (all being provable in $\mathscr{E}$, i.e., $\mathscr{F}{}^*\mathscr{E} \subseteq \mathscr{E} = {}^*\mathscr{E}$). Hence, if $\mathscr{E}$ is very hard, then $\mathscr{F}{}^*\mathscr{E} \subsetneq \mathscr{E}$, and thus completeness of $\mathscr{E}$ implies incompleteness of $\mathscr{F}{}^*\mathscr{E}$. Therefore we have:

INCOMPLETENESS THEOREM 6.10. *If $\mathscr{E}$ is any of the systems mentioned by Theorems* 6.3 *and* 6.6, *then $\mathscr{F}*\mathscr{E}$ is incomplete* (*i.e.,* $\mathscr{F}*\mathscr{E}$ *does not prove all tautologies*).

Regarding each of these incomplete finitary systems $\mathscr{F}*\mathscr{E}$, there are two, somewhat conceptual open problems:

Problems 6.11. ·) to find a formula A such that $\mathscr{E} \vdash A$ & $\mathscr{F}*\mathscr{E} \nvdash A$ (such formulae obviously exist)

·) to find (or to disprove the existence of) a class of models $\mathscr{A}$ such that $\mathscr{F}*\mathscr{E}$ is complete with respect to $\mathscr{A}$.

An instruction to these problems might possibly be found in Wand's and Clarke's examples of formulae whose proofs require ω-rules, cf. Goldblatt's exposé on the matter (Goldblatt, 1982, 1.6, 1.10, 3.10, and p. 191).

·) to find some clearer description of the systems in which the (Cov)-type rules are redundant; see Theorem I.7.8.

Chapter III: Quantification in Combinatory PDL: CDL

> The novice may wonder why quantified modal logic (QML) is considered difficult. QML would seem to be easy: simply add the principles of first-order logic to propositional modal logic. Unfortunately, this choice does not correspond to an intuitively satisfying semantics. From the semantical point of view, we are confronted with a number of decisions concerning the quantifiers, and these in turn prompt new questions about the semantics of identity, terms and predicates. Since most of the choices can be made independently, the number of interesting quantified modal logics seems bewilderingly large.

This is the beginning of Garson's chapter (Garson, 1984) of the "Handbook of Philosophical Logic," later in which p. 269, Garson writes:

> Completeness proofs in QML are quite a bit harder than the completeness proofs for propositional modal logic or first order logic. One reason that proofs are difficult is that sometimes there are none to find.... Even when a system is complete, the proof may be elusive, and difficult to formulate in a simple way. Another problem is lack of generality: a proof strategy may only work when the underlying modal logic is fairly strong, or when *ad hoc* conditions are placed on the models.
>
> One of the best ways to understand the methods used in completeness proofs for QML is to locate the main difficulty which arises if we simply try to 'paste together' proofs for quantificational logic and propositional modal logic.

To add to this sketch of modal predicate troubles, we shall also quote van Benthem's Chapter [Benthem, 1984, Sect. 2.5] of the same Handbook:

> Modal *predicate* logic, however important in philosophical applications, is much less understood.... The unfinished state of the art shows already in the fact that no commonly accepted notion of semantic structure, or truth definition exists.... On the whole, exciting technical results are yet scarce in modal predicate logic.

These quotations (and we can add many more) witness to the disastrous situation of modal predicate logic from its pre-combinatory period. The difficulties (most of, if not all) in quantified modal logic seem to be caused by the two different sorts of quantification: one modal $[R]$, and one classical $\forall x$ or $\forall p$. For, in search of completeness, the main concern becomes, cf. (Garson, 1984, Sect. 2.1.3), to co-ordinate the conditions of the Truth lemma, demanded by the two quantifiers.

These problems are to be skirted, if classical quantification is introduced to match modal one. Since the latter is over possible worlds, $s \models [R]A$ iff $\forall t(sRt \rightarrow t \models A)$, we should be obliged to classically quantify over the worlds as well. Having the names at hand, we shall have no problems in doing so: we add the syntactical clause "$\forall cA$ is a formula" with the most, if not the only, natural semantics and axiomatics.

The troubles are now overcome. The logic obtained is easy: easy to understand and easy to deal with. An exciting completeness result is obtained. And the novice will have one less point to wonder about.

Technically this chapter might be considered as a part of the previous one (the notational and the like conventions being still valid), and we artificially separate the quantified study only because of the complications arising in the non-combinatory case.

The idea to introduce quantifiers in combinatory language came from Skordev's paper (Skordev, 1984), which suggests quantification (in a rather computational system) over objects similar to what we denote by $\hat{c}$.

1. *Definition and Completeness*

Extending the definition of CPDL, we set:

Syntax of CDL: $\exists cA$ is a formula, and $\forall cA =_{\mathrm{DF}} \neg\exists c \neg A$.

By $\{{}^c_d\}A$ we shall denote the substitution of c instead of d in A, assuming c to be *substitutable* for d in A, cf. (Shoenfield, 1967). The notion of *free* occurrence of a name in a formula or program is the usual one.

Semantics of CDL: $s \models \exists cA$ iff $s \models \{{}^d_c\}A$, for some $d \in \Sigma$.

Axiomatics of CDL: We discard (Cov) and add to CPDL's axiomatics:

(cov) $\exists cc$
($\forall -$ax) $\forall cA \rightarrow \{{}^{d}_{c}\} A$
(barc) $\forall c[\alpha] A \rightarrow [\alpha] \forall cA$, if c has no free occurrence in α (Barcan formula),

and the rule

(Gen) If $\vdash A$, then $\vdash \forall cA$.

Query. Is the Barcan formula derivable? (Bull (1970) and Prior (1967) note that it is, when $[\alpha]$ is $S5$, or when $[\alpha^{-1}]$ exists in the language.)

LEMMA 1.1. *The rule* (Cov) *is admissible.*

Proof. Let $\vdash [\gamma] \neg c$, for all $c \in \Sigma$. Let $c \not\alpha \gamma$.

$\vdash [\gamma] \forall c \neg c$, by (Gen) and (barc)

$\vdash [\gamma] \forall cc$, by (cov) and (Nec)

$\vdash [\gamma] 0$, by the last two. ∎

LEMMA 1.2. *The following "special" rule is admissible*:

(SR) *If* $[\gamma]\{{}^{d}_{c}\} A$, *for each* $d \in \Sigma$, *then* $\vdash [\gamma] \forall cA$.

Proof. By (Gen) and (barc). ∎

DEFINITION 1.3. A *logic* over CDL is any set of CDL formulae, containing all axioms and closed under (Ind), (MP), (Nec), and (Gen). Given a logic L, an L*-theory* over CDL is a set containing L and closed under (Ind), (MP), and (SR).

Note that (SR)-closedness of theories guarantees (Cov)-closedness as well.

TRUTH LEMMA 1.4. *We extend the Truth lemma* I.5.1 *with the following clause*:

$$\textcircled{c}\, \forall dA \in T \quad \textit{iff} \quad \textcircled{c}\, \{{}^{e}_{d}\} A \in T, \textit{for all } e \in \Sigma.$$

Proof. By (SR)-closedness of T and ($\forall -$ax). ∎

The familiar canonical technique gives:

THEOREM 1.5. *CDL is complete.*

2. *Arithmetical Operators: Axiomatization through Expressiveness*

The domain of the translation $\tilde{}$ (cf. I.1.8) can be uniformly extended from open to all PC formulae; thus obtained will be an injective imbedding $\tilde{}$: PC $\mapsto$ CDL. We can extend the Expressiveness theorem 1.9 to the following

EXPRESSIVENESS THEOREM 2.1. *For each PC formula F, $F^{\sim}$ expresses F.*

Proof. Straightforward induction on the construction of F. ▮

Questions 2.2. ·) How to extend PC to equalize its expressive power with CDL? (Such an extension should contain a sort of transitive-closure-operator, cf. the remarks preceding Problem I.1.13.)

·) Describe the set $\{A \in \text{CDL}/A$ expresses some PC formula$\}$.

Let us fix, cf. Remark II.2.2, an arbitrary PC formula $F(c, d)$ of two free variables and extend CDL with the operator $\theta = \theta_F(\boldsymbol{\alpha})$, adding the clauses:

Syntax of CDL$^{\theta}$: $\theta(\boldsymbol{\alpha})$ is a program.

Semantics: $\chi(c)\, R_\theta \chi(d)$ iff $\mathcal{M} \Vdash F(c, d)$.

THEOREM 2.3. *Let $A(c, d)$ express $F(c, d)$. Then*

$$(\mathrm{Ax}_\theta) \quad \text{ⓒ}\langle\theta(\boldsymbol{\alpha})\rangle d \leftrightarrow A(c, d)$$

completely axiomatizes CDL$^{\theta}$.

Proof. We mechanically extend the completeness proof for CDL with the details listed in the proof of Theorem II.2.3. ▮

As a corollary to Theorems 2.1 and 2.3, we obtain

THEOREM 2.4. *Let $F(c, d)$ be an arbitrary PC formula, and $\theta = \theta_F$. Then*

$$(\mathrm{Ax}_\theta) \quad \text{ⓒ}\langle\theta(\boldsymbol{\alpha})\rangle d \leftrightarrow F^{\sim}(c, d)$$

completely axiomatizes CDL$^{\theta}$.

Problems 2.5. It is important for the last proof that F be expressible, not that F be from PC. So, having once answered Question 2.2, one immediately gets a generalization of Theorem 2.4 covering each F from the superstructure of PC sought expressible in CDL. An accomplishment of

Theorem 2.4 would require an explicit description of all operators, axiomatizable over CDL by a finite set of axioms. Also open is the question of what a breakthrough in the analytical hierarchy the combinatory axiomatic potential can make. For we have one example with the repeating Δ, whose axiomatization makes hints to introducing longer or *quasi-longer* rules, and the harder operators might possibly require giving names to the subsets of the universe: such an idea might envelope Bull's (1970) *history-propositional variables* in tense logic, and Radev's (1986) *path constants* in process logic, both nominating strings of states.

We should note that process logics offer expressiveness (or descriptive completeness) results similar to that of Theorem 2.1; see (Nishimura, 1980) and references therein.

3. *Undecidability and Finitary Incompleteness*

Since $\{F^{\sim}/F \in \mathrm{PC}\}$ is a decidable subset of CDL formulas, and $\Vdash F$ iff $\mathrm{PC} \vdash F$, we have the PC-provability problem reducible to the CDL-provability problem, and in virtue of the Σ_1^0-completeness of the former, one has:

PROPOSITION 3.1. *Deciding provability in CDL is at least Σ_1^0-complete.*

On the semantical side, one has

PROPOSITION 3.2. *The language of CDL lacks the fmp.*

Proof. The formula $\boxdot\langle a\rangle 1 \wedge \forall c \,\textcircled{\diamond}\, [a; a^*] \neg c$ has only infinite models. ▮

We shall strengthen these results. As we mentioned in Section II.6, the essence of the high undecidability of Deterministic $\mathrm{PDL}^{\cap}$ is in the guaranteed interpretation of two programs a, b as deterministic, and in the existence of a formula (with the semantics of) $\langle ab \cap ba\rangle 1$. In CDL's language, we have that

$$\det(a) =_{\mathrm{DF}} \boxdot\, \forall c(\langle a\rangle c \rightarrow [a]c), \qquad \text{and likewise} \qquad \det(b),$$

guarantee determinism of a and b, and that

$$\exists c(\langle ab\rangle c \wedge \langle ba\rangle c) \qquad \text{has the meaning of } \langle ab \cap ba\rangle 1.$$

So, again as a corollary to (Harel, 1983, Theorem 4.9), we have

THEOREM 3.3. *CDL is highly undecidable.*

Hence we can extend the incompleteness results from II.6.10 with

INCOMPLETENESS THEOREM 3.4. *$\mathscr{F}$ CDL is not complete.*

By the arguments of Proposition 3.1, and the recursive enumerability of $\mathscr{F}$CDL theorems, we have

THEOREM 3.5. *Deciding provability in $\mathscr{F}$ CDL is Σ_1^0-complete.*

As a by-product of the method, we obtain for the "pure," *-free Quantified Combinatory poly-modal logic, CpML:

THEOREM 3.6. *CpML is Kripke complete and Σ_1^0-complete.*

The incompleteness again raises problems (see the instruction to II.6.11):

Problems 3.7. ·) find a formula A such that $\mathscr{F}$CDL $\not\vdash A$ & CDL $\vdash A$.

·) find (or disprove the existence of) semantics w.r.t. which $\mathscr{F}$CDL is complete.

Discussion on the State of the Art, Its Perspectives and Genesis

The truly-Henkin combinatory completeness proof, unlike the traditional quasi-Henkin proofs of modal completeness, provides a very flexible counter-model fitting for a wide family of extensions (and proceeds without making a detour through non-standard models or pseudomodels). The basic language succeeds to impose on the counter-model all the conditions demanded by the semantics of the various operations that we mercilessly added to the syntax. And not surprisingly: "In some sense this logic contains its own metatheory," cf. (Radev, 1987). The completeness proof as if emerges from the coherence between the combinatory syntax and models. Proper names provide a rich syntactic matter out of which the counter-model grows, exactly as the variable-free terms serve in the predicate calculus, cf. (Shoenfield, 1967).

Moreover, the rule (Cov), respectively the predicate axiom (cov), allots to the names the role of special Henkin constants, making them witnesses to the modal existence $\langle v \rangle$. For the Truth lemma I.5.1 (4) guarantees that for each formula $A = \langle \alpha \rangle B$, there is a name c_A with $T \vdash A \rightarrow \langle \alpha \rangle (c_A \wedge B)$, which precisely responds to the generic condition for the special Henkin constant c_A for the formula $A = \exists x B(x)$: $\vdash A \rightarrow B(c_A)$.

Similarly, in the quantified case, the axiom (cov) makes the names witnesses also to classical existence $\exists$. So, no obstacles exist to mechanically "pasting together" the Henkin proof for predicate calculus and the Henkin proof for the Combinatory PDL into a smooth completeness proof for CDL.

The charm of CDL might be put in more ceremonial words, saying that CDL is a *free logic* in the sense of Hintikka, see, e.g., (Garson, 1984). This means that the existential quantifier $\exists$ refers only to objects whose existence is guaranteed (namely by the axiom $\langle v \rangle c$). Hence, the choice of CDL as a predicate model system accords with Garson's "Conclusion: We Should Adopt Free Logic."

As far as the ω-rule (Ind) is concerned, we can not but accept Goldblatt's (1982, p. 24) contention that "reasoning about *while*-commands is naturally encapsulated in an infinitary rule, and so such rules are to be used in a system that is designed to formally represent such reasoning." In fact, (Ind) makes the natural numbers witnesses to the implicit quantifier $\exists n$ realized by the iteration *: the Truth lemma says that, for a formula $A = \langle \alpha^* \rangle B$, there is a natural number k with $T \vdash A \rightarrow \langle \alpha^k \rangle B$.

As shown by Proposition II.6.8, the (Cov)-type infinity is, as a rule, a quasi-infinity, and thus is of the two ω-evils the lesser. The Incompleteness theorems II.6.10 and III.3.4, on the other hand, show for many systems that finitary axiomatization, hence (Ind)'s finitization, is impossible, whence (Ind) remains the carrier of actual infinity. This rule is a means justified by the end: uniform axiomatization of the variety of dynamic modal systems—from easily decidable to highly undecidable.

The high undecidability of the combinatory systems, in turn, is a projection of the high undecidability of a certain non-combinatory counterpart, cf. Theorems II.6.3, II.6.6, III.3.3. So the very hardness is properly due not to the combinatory innovations, but rather to the state of affairs, which state is only accurately depicted, and not caused, by the combinatory approach which we professed in this paper.

For future research, directions were outlined in I.1, I.1.13, II.2.4, II.2.8, II.2.20, II.3.8, II.5, II.6.7, II.6.11, III.2.2, III.2.5 and III.3.7. Well-motivated, as a possible alternative of the ω-rules, are versions of the combinatory language admitting infinitary conjunctions and disjunctions. Progress in that direction has been made by Radev (Radev, 1985, 1987), and future research should possibly be linked with axiomatization through (Σ, Δ, $\Sigma\Delta$)-epressiveness, cf. II.2.20.

Another perspective is the nominated process logic, and here also a pioneering work exists by Radev (1986). Let us mention also temporal, tense, intuitionistic, and epistemic logics; from the latter we shall distinguish some knowledge representation systems, cf. (Halpern, 1986), in which Kripke semantics and $S5$ are so fashionable today. On the whole, the combinatory idea suggests a revision, though a slight one, of the fundamentals of Kripke modal logic, and therefore benefits should be expected in all areas which Kripke semantics has to do with.

A good idea should enjoy a respectable history. Our names have it. We traced this history back to the mid-fifties, when they were probably conceived.

Prior (1956) launches the idea of *variables ranging over individual moments of time*, and Bull (1968) proves this system complete. Reportedly, cf. Humberstone (1987b), Prior and Meredith (1965) and Prior and Fine (1977) consider objects similar to what we call names.

Fine (1970) axiomatizes *Boolean atoms* via propositional quantifiers over *S*5; he reports on a similar result of David Kaplan. Later, Fine (1975) introduces the so-called *normal forms* to serve as *state-descriptions*, and provides elegant completeness proofs.

Gabbay (1976) mentions *proper names* in a temporal context. Garson (1980) quantifies over *individual concepts* (= functions from possible worlds to possible worlds). Gabbay (1981) suggests a syntactical modal characterization of irreflexive accessibility relation R_a, by the rule

$$\text{“If } \vdash [\gamma]([a]p \rightarrow p), \qquad \text{then} \quad \vdash [\gamma]0\text{,”}$$

where $[\gamma]$ is an admissible form, cf. Comment I.3.5.D. He writes: "This rule simply allows us to carry out a semantic tableaux construction for the logic, allowing us to build models with the property that in each possible world there is an atom {i.e., variable} true exactly in that world."

The normal forms of (Fine, 1975) were rediscovered a decate later by Fagin and Vardi (1985), for proving completeness w.r.t. the *internal modal semantics*—an approach also closely related to the names, cf. (Tehlikeli, 1985).

Tiomkin and Makowksi (1985) introduce *local assignments* over PDL—a semantical notion close to the names but less expressive, cf. (Tinchev, 1988). A similar *unique-world modality*, also a purely semantical creature, appears in (Koymans, 1988).

An axiomatic approach to dynamic logic close to the combinatory spirit is that of Goldblatt (1982, 1987). He clearly declares that "data have to be nameable in programming language" (1982, p. 111), and that "infinitary rule cannot be replaced..." (1982, p. 170). These views lead the author to a truly-Henkin completeness proof of a quantificational dynamic logic (1987), and to a version of Scott's isomorphism theorem (1982, p. 1974, Corollary 3.7.3). In contrast with our approach, Goldblatt adjoins to the language some external set of technical witnesses, in order to obtain *rich theories*, namely theories closed under a rule similar to what we call (SR) in Chapter III. The external witnesses, as compared to the proper names, have both advantages and drawbacks in both technical and conceptual respects, and we leave deeper comparisons to the future.

A paper presently belonging rather to the future than to the history of the subject is that of Blackburn (1989a) (and Blackburn, 1989b), referred to therein), who rediscovers names in the context of tense logic. We were happy to learn that this author had survived the same diappointment as we ourselves had; he writes: "When I began the work reported here I believed the idea of using nominals to be a novel one."

A paper, however, evidently holding priority over introducing names in their cleanest form is that of Bull (Bull, 1970), where names, under the name of *clock-propositional variables*, appear in the context of a quantificational tense logic with a universe-modality, the quantification being namely over them. (Bull refers to Arthur Prior (1967) and Bull (1989) notes "The idea was Arthur's in the first place.") Though independent (we saw Bull's and Prior's works in the summer of 1989), the present study might be considered as an advance of Prior and Bull's idea.

By happenstance, Kripke modal theory missed the chance of being *ab initio* created as a combinatory one: the idea of names can be uncovered as a divide between the two historical modal roots—(Kripke, 1959) and (Kripke, 1963). In the latter Kripke writes:

> Every atomic formula {i.e., propositional variable} P is assigned a truth-value in each world H; in fact, this truth-value is $\phi(P, H)$. Here we already have a slight divergence from the treatment in [1959]. For in [1959], we did not have an auxiliary function ϕ to assign truth-value to P in the world H; instead H itself was a "complete assignment," that is, a *function* assigning a truth-value to every atomic subformula of a formula A. On this definition, "worlds" and complete assignments are identified; so distinct worlds give distinct complete assignments. This last clause means that *there can be no two worlds in which the same truth-value is assigned to each atomic formula.* Now this assumption turns out to be convenient perhaps for $S5$, but it is rather inconvenient when we treat normal MPC's in general. In the present paper we drop it.

A feature distinctive for the Combinatory approach is that, bringing the modal language closer to the meta-language, an equilibrium is kept between the three pillars of abstractness: language—with names and universe-modality, semantics—with possible worlds and universe, and axiomatics—with axioms fixing what is possible and necessary to be fixed.

APPENDIX: Stone Representation Theorem for Combinatory Dynamic Algebras

DEFINITION 7.1. The tuple $D = ((\mathbf{B}, \mathscr{E}, 0, \wedge, \neg), (\mathbf{P}, \nu, ;, \cup, {}^*), [\ ], ?)$ is called a *Combinatory Dynamic Algebra*, CDA, if

(i) $(B, 0, \wedge, \neg)$ is a Boolean algebra, and

(ii) $\mathscr{E} \subseteq \mathbf{B}$, $\nu \in \mathbf{P}$, $[\ ]$: $\mathbf{P} \times \mathbf{B} \to \mathbf{B}$, $?$: $\mathbf{B} \to \mathbf{P}$ satisfy the following conditions (where $A, B \in \mathbf{B}$, α, β, $\gamma \in \mathbf{P}$, $c \in \mathscr{E}$, $\langle\ \rangle =_{\mathrm{DF}} \neg[\ ]\neg$, $\boxdot =_{\mathrm{DF}} [v]$, $\boxed{c} =_{\mathrm{DF}} [v; c?]$, $\langle c \rangle =_{\mathrm{DF}} \neg \boxed{c} \neg$):

$$[\alpha](A \wedge B) = [\alpha]A \wedge [\alpha]B$$

$$\boxdot 1 = 1$$

$$[\alpha \cup \beta]A = [\alpha]A \wedge [\beta]A$$

$$[\alpha\beta]A = [\alpha][\beta]A$$

$$[A?]B = A \to B$$

$$\boxdot A \leqslant A,\ \boxdot A \leqslant \boxdot \boxdot A,\ A \leqslant \boxdot \diamond A$$

$$\boxdot A \leqslant [\alpha]A$$

$$\langle c \rangle A = \boxed{c} A$$

$$\bigwedge_{n<\omega} [\gamma][\alpha^n]A = [\gamma][\alpha^*]A$$

$$\bigwedge_{c \in \Sigma} [\gamma] \neg c = [\gamma]0.$$

Let $\{D_k\}_{k \in I}$ be some family of CDA's. We define the Cartesian product ΠD_k in the standard way and prove it to be a CDA as well. Following (Pratt, 1979), we say that D is *separable*, if $\alpha \neq \beta$ implies $\exists A$ $(\langle\alpha\rangle A \neq \langle\beta\rangle A)$. (Note that D is separable iff $\alpha \neq \beta$ implies $\exists c$ $(\langle\alpha\rangle c \neq \langle\beta\rangle c)$.) We say that D is a *set-CDA*, if for some set X it is the case that $\mathbf{B} = 2^X$, $\mathscr{E} = \{\{s\}/s \in X\}$, $\mathbf{P} = 2^{X \times X}$, $v = X^2$, and the remaining are the usual dynamic operations.

STONE REPRESENTATION THEOREM 7.2. *Let D be a countable, separable CDA. Then there is a family $\{D_k\}$ of set-CDAs and an isomorphism h: $D \to \Pi D_k$.*

Proof. Cf. (Tinchev, 1986). ∎

Note. In the non-combinatory version of this theorem, see (Vakarelov, 1983), D is required to be, besides separable, also free in the class of all separable DA's.

If we extend the definition of CDA by a complement operation $\neg$ in $\mathbf{P}$ and the equality $\langle\neg\alpha\rangle c = [\alpha] \neg c$, the Stone theorem will still hold.

This appears to solve the problem raised by Pratt (1979) for creating a notion of "complemented dynamic algebra." It might be tempting, moreover, to create a notion of Quantified CDA, in the spirit of Chapter III, and prove a Representation theorem, thus trying to challenge van Benthem's (1984, p. 217) statement that "elegant algebraization stops at the gates of predicate logic."

Bibliographical Appendix

To the titles and names hitherto mentioned, we should add at least a few more, concerning some of the particular topics.

For the intersection of programs: Harel, Pnueli, and Vardi, cf. (Harel, 1984), Harel and Vardi, cf. (Harel, 1983), Fariñas del Cerro and Orlowska (1985), Humberstone (1985), Gargov (1986), Gargov and Passy (1990), and Vakarelov (1985–1988).

For the complement of programs: Harel (1984), who is also a source for converse, cycle, repeat, the iteration quantifier, and other things.

For the universe-modality: Goranko and Passy (1990).

For the bounded non-determinism: Mirkowska (1981) and Mirkowska and Salwicki (1987).

For the iteration quantifier: Salwicki (1970), Mirkowska (1981), Goldblatt (1982), Harel (1984), and Mirkowska & Salwicki (1987).

For the ω-rule in PDL: Goldblatt (1982, 1987), Harel (1984), and Tinchev and Vakarelov (1986); and for the ω-rules in classical logic we refer to Rasiowa and Sikorski (1963) and Sundholm (1978).

The idea of possible worlds semantics for modal logic, today known as Kripke and/or Hintikka semantics, is also attributed to mid-forties papers of Carnap, and, if the will is there, may be found even in scripts of Leibniz, see (Mates, 1968). Starting a decade after Carnap, pioneering in the development of the idea were A. Bayart, E. Beth, M. Guillaume, J. Hintikka, R. Montague, S. Kanger, S. Kripke, A. Prior, until it crystalyzed in (Kripke, 1963). (For more detailed historical references we point to Fagin and Vardi (1985) and Hintikka (1984).) There are even debates on priority over this celebrated approach, and naming it after Kripke we follow the tradition obeyed also by the other pioneers, cf. Hintikka (1984).

When speaking of history of and priority in modal logic two more references should not be omitted. One is that of Bull and Segerberg (1984), which is an invaluable document for the development of the art. The other one is qualified in the latter: "A particularly interesting paper with implications for modal logic is Jónsson and Tarski (1951). If it had been widely read when it was published, the history of modal logic might have looked different." The history of modal logic has lost this chance (as well as many others), but this might serve as an instructive example for the future of the art, which might look different, if the present paper is not widely read when published.

This sketch of references is by no means complete, let alone detailed. However, as a full list would, we fear, utterly exhaust both the authors and—a more immediate concern to them—the patient reader, we will here bring this to its final point.

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References

AHO, A., AND ULLMAN, J. (1979), Universality of data retrivial languages, *in* "Proceedings of 6th ACM Symposium on Principles of Programming Languages," pp. 110–117.

ANDRÉKA, H., NÉMETI, I., AND SAIN, I. (1982), A complete logic for reasoning about programs via non-standard model theory, I and II, *Theoret. Comput. Sci.* **17**, 193–212 and 259–278.

BEN-ARI, M., HALPERN, J., AND PNUELI, A. (1982), Deterministic propositional dynamic logic: Finite models, complexity, and completeness, *J. Comput. System Sci.* **25**, 402–417.

VAN BENTHEM, J. F. A. K. (1979), Minimal deontic logics [abstract], *Bull. Sec. Logic* **8**, 1 (March), 36–42.

VAN BENTHEM, J. F. A. K. (1984), Correspondence theory, *in* [Gabbay and Guenthner, 1984], Vol. II, pp. 167–247.

BLACKBURN, P. (1989a), "Nominal Tense Logic," Centre for Cognitive Science, University of Edinburgh.

BLACKBURN, P. (1989b), "Reasoning in and about Time: Nominal Tense Logic and Other Sorted Intensional Frameworks," Ph.D. Thesis, Centre for Cognitive Science, University of Edinburgh.

BULL, R. (1968), On possible worlds in propositional calculi, *Theoria* **34**, 3, 171–182.

BULL, R. (1970), An approach to tense logic, *Theoria* **36**, 3, 282–300.

BULL, R. (1989), private communication.

BULL, R., AND SEGERBERG, K. (1984), Basic modal logic, in [Gabbay and Guenthner, 1984], Vol. II, pp. 1–88.

DANECKI, R. (1985), Nondeterministic propositional dynamic logic with intersection is decidable, *in* "Lecture Notes in Computer Science," Vol. 208, pp. 34–53, Springer-Verlag, Berlin/New York.

DOŠEN, K. (1986), Modal duality theory, *in* "Proc. of the Conf. *Algebra and Logic*," pp. 73–88, Cetinje.

FAGIN, R., AND VARDI, M. (1985), "An Internal Semantics for Modal Logic: Preliminary Report," Report CSLI-85-25, Stanford University.

FARIÑAS DER CERRO, L., AND ORLOWSKA, E. (1985), DAL—A logic for data analysis, *Theoret. Comput. Sci.* **35**, 251–264.

FATTOROSI-BARNABA, M., AND DE CARRO, F. (1985), Graded modalities, I, *Studia Logica* **44**, 197–221.

FINE, K. (1970), Propositional quantifiers in modal logic, *Theoria* **36**, 336–346.

FINE, K. (1975), Normal forms in modal logic, *Notre Dame J. Formal Logic* **16**, 229–237.

FISCHER, M., AND LADNER, R. (1979), Propositional dynamic logic of regular programs, *J. Comput. System. Sci.* **18**, 194–211, earlier version *in* "Proc. 9th Ann. ACM STOC (1977)," pp. 286–294.

GABBAY, D. (1976), "Investigations in Modal and Tense Logic with Applications to Problems in Philosophy and Linguistics," Reidel, Dordrecht.

GABBAY, D. (1981), An irreflexivity lemma with applications to axiomatizations of conditions on tense frames, *in* "Aspects of Philosophical Logic" (U. Moenich, Ed.), pp. 67–89, Reidel, Dordrecht.

GABBAY, D., AND GUENTHNER, F., Eds. (1984), "Handbook of Philosophical Logic," Vols. 1–4, Reidel, Dordrecht.

GARGOV, G. (1984–1986), oral communication and unpublished manuscripts.

GARGOV, G. (1985), Decidability of the basic combinatory propositional dynamic logic, *in* "Mathematical Theory of Programming" (A. Ershov and D. Skordev, Eds.), Computer Centre of the Siberian Division of Soviet Academy of Sciences, Novosibirsk.

GARGOV, G. (1986), Two completeness theorems in the logic of data analysis, ICS PAS Report 581 (April), Warsaw.

GARGOV, G., AND PASSY, S. (1985), A μ-calculus based on combinatory PDL, manuscript.

GARGOV, G., AND PASSY, S. (1988), Determinism and looping in combinatory PDL, *Theoret. Comput. Sci.* **61**, 259–277.

GARGOV, G., AND PASSY, S. (1989), A note on Boolean modal logic, *in* [P. Petkov, 1991].

GARGOV, G., PASSY, S., AND TINCHEV, T. (1987), Modal environment for Boolean speculations (preliminary report), *in* [Skordev, 1987], pp. 253–263.

GARSON, J. (1980), The unaxiomatizability of a quantified intensional logic, *J. Philos. Logic* **9**.

GARSON, J. (1984), Quantification in modal logic, *in* [Gabbay and Guenthner, 1984], Vol. II, pp. 249–307.

GOLDBLATT, R. (1982), Axiomatizing the logic of computer programming, *in* "Lecture Notes in Computer Science," Vol. 13, Springer-Verlag, Berlin/New York.

GOLDBLATT, R. (1987), "Logics of Time and Computation," CSLI Lecture Notes, # 7.

GOLDBLATT, R., AND THOMASON, S. (1975), Axiomatic classes in propositional modal logic, *in* "Lecture Notes in Mathematics," Vol. 450, pp. 163–173, Springer-Verlag, Berlin/New York.

GORANKO, V. (1987), Modal definability in enriched languages, *Notre Dame J. Formal Logic*, to appear.

GORANKO, V. (1989), Completeness and incompleteness in the Bi-modal base $\mathscr{L}(R, -R)$, *in* [Petkov, P., 1991].

GORANKO, V., AND PASSY, S. (1990), Using the universal modality: Gains and questions, University of Amsterdam, ITLI Prepublication Series, X-90-06.

HABASINSKI, Z. (1985), Model theory of propositional logics of programs. Some open problems, *in* "Lecture Notes in Computer Science," Vol. 208, pp. 98–110, Springer-Verlag, Berlin/New York.

HALPERN, J., Ed. (1986), Theoretical aspects of reasoning about knowledge, *in* "Proc. of the 1986 Conf.," Morgan Kaufman.

HALPERN, J., AND REIF, J. (1983), The propositional dynamic logic of deterministic, well-structured programs, *Theoret. Comput. Sci.* **27**, 127–165.

HAREL, D. (1979), "First-Order Dynamic Logic," Lecture Notes in Computer Science, Vol. 68, Springer-Verlag, Berlin/New York.

HAREL, D. (1983), Recurring dominoes: Making the highly undecidable highly understandable, *in* "Lecture Notes in Computer Science," Vol. 158, pp. 177–194, Springer-Verlag, Berlin/New York, and *Ann. Discrete Math.* **24** (1985), 51–72.

HAREL, D. (1984), Dynamic logic, *in* [Gabbay and Guenthner, 1984], Vol. II, pp. 605–714.

HINTIKKA, J. (1984), Is alethic modal logic possible? in "Modal and Intensional Logics" (V. A. Smirnov, Ed.), pp. 31–49, Nauka, Moscow.

HUMBERSTONE, I. L. (1983), Inaccessible worlds, *Notre Dame J. Formal Logic* **24**, 3 (July), 346–352.

HUMBERSTONE, I. L. (1985), The formalities of collective omniscience, *Philos. Studies* **48**, 401–423.

HUMBERSTONE, I. L. (1987a), The modal logic of 'all and only', *Notre Dame J. Formal Logic* **28**, 2, 177–188.

HUMBERSTONE, I. L. (1987b), private communication, June.

JÓNSSON, E., AND TARSKI, A. (1951), Boolean algebras with operators, I, *Amer. J. Math.* **73**, 891–939.

KOYMANS, R. (1988), Extending modal and temporal logics with inequality and metric operators, manuscript, Eindhoven.

KOZEN, D., AND PARIKH, R. (1981), An elementary proof of the completeness of PDL, *Theoret. Comput. Sci.* **14**, 113–118.

KOZEN, D., AND TIURYN, J. (1987), "Logics of Programs," Computer Science Dept., Washington State Univ., CS-87-172.

KRIPKE, S. (1959), A completeness theorem in modal logic, *J. Symbolic Logic* **24**, 1–14.

KRIPKE, S. (1963), Semantic analysis of modal logic, I: Normal propositional calculi, *Z. Math. Logik Grundl. Math.* **9**, 67–96.

MATES, B. (1968), Leibniz on possible worlds, *in* "Logic Methodology and Philosophy of Science III, Proc. of the 3rd International Congress, Amsterdam, 1967," (B. van Roostelaar and J. Staal, Eds.), pp. 507–529, North-Holland, Amsterdam.

MEYER, A., STREETT, R., AND MIRKOWSKA, G. (1981), The deducibility problem in propositional dynamic logic, *in* "Lecture Notes in Computer Science," Vol. 115, pp. 238–248, Springer-Verlag, Berlin/New York; "Lecture Notes in Computer Science," Vol. 125, pp. 12–22, Springer-Verlag, Berlin/New York.

MIRKOWSKA, G. (1981), PAL—Propositional algorithmic logic, *in* "Lecture Notes in Computer Science," Vol. 125, p. 23–101, Springer-Verlag, Berlini/New York.

MIRKOWSKA, G., AND SALWICKI, A. (1987), "Algorithmic Logic," PWN and Reidel.

NISHIMURA, H. (1980), Descriptively complete process logic, *Acta Inform.* **14**, 359–369.

PARIKH, R. (1978), "A Completeness Result for Propositional Dynamic Logic," MIT/LCS/TM-106.

PARIKH, R. (1981), Propositional dynamic logics of programs: A survey, *in* "Lecture Notes in Computer Science," Vol. 125, pp. 102–144, Springer-Verlag, Berliin/New York.

PARIKH, R. (1983), Propositional logics of programs: New directions, *in* "Lecture Notes in Computer Science," Vol. 158, pp. 347–359, Springer-Verlag, Berlin/New York.

PASSY, S. (1984), "Combinatory Dynamic Logic," Ph.D. Thesis, Faculty of Mathematics, Sofia University.

PASSY, S., AND TINCHEV, T. (1984), Data constants in PDL: The natural state of affairs, unpublished.

PASSY, S., AND TINCHEV, T. (1985a), PDL with data constants, *Inform. Process. Lett.* **20**, 35–41.

PASSY, S., AND TINCHEV, T. (1985b), Quantifiers in combinatory PDL: Completeness, definability, incompleteness, *in* "Lecture Notes in Computer Science," Vol. 199, pp. 512–519, Springer-Verlag, Berlin/New York.

PELEG, D. (1987), Concurrent dynamic logic, *J. Assoc. Comput. Mach.* **34**, 450–479.

PETKOV, A. (1987), Propositional dynamic logic in two and more dimensions, *in* [Skordev, 1987], pp. 323–329.

PETKOV, P., Ed. (1991), "Mathematical Logic, Proceedings of the Summer School and Conference dedicated to the 90th anniversary of Arend Heyting, Chaika, 1988," Plenum, New York/London.

PRATT, V. (1976), Semantical considerations on Floyd–Hoare logic, *in* "Proc. 17th Ann. Sympos., on FOCS," pp. 109–121.

PRATT, V. (1979), "Dynamic Algebras: Examples, Constructions, Applications," MIT/LCS/TM-138.

PRIOR, A. (1956), Modality and quantification in *S5*, *J. Symbolic Logic* **21**, 60–62.

PRIOR, A. (1967), "Past, Present, and Future," Oxford Univ. Press.

PRIOR, A., AND FINE, K. (1977), "Worlds, Times, and Selves," Univ. of Massachsetts Press, Amherst.

PRIOR, A., AND MEREDITH (1965), *Notre-Dame J. Formal Logic.*

RADEV, S. (1985), Extension of PDL and consequence relations, *in* "Lecture Notes in Computer Science," Vol. 208, pp. 251–264, Springer-Verlag, Berlin/New York.

RADEV, S. (1986), Process logic with path constants, manuscript.

RADEV, S. (1987), Infinitary propositional normal modal logic, *Studia Logica* **46** (4), 291–309.

RASIOWA, H. (1983), A talk at the Summer School on Mathematical Logic and its applications, Primorsko, unpublished.

RASIOWA, H., AND SIKORSKI, R. (1963), "Mathematics of Metamathematics," PWN, Warsaw.

ROGERS JR., H. (1967), "Theory of Recursive Functions and Effective Computability," McGraw–Hill, New York.

SAKALAUSKAITE, J., AND VALIEV, M. (1989), Completeness of propositional dynamic logic with infinite repeating, to appear *in* [P. Petkov, 1989].

SALWICKI, A. (1970), Formalized algorithmic languages, *Bull. Acad. Pol. Sci., Ser. Sci. Math., Astr. etc.* **18**, 277–332.

SALWICKI, A. (1987), Oral communication.

SCOTT, D. (1963), A logic with denumerable long formulas and finite strings of quantifiers, *in* "The Theory of Models" (J. Addison, L. Henkin, and A. Tarski, Eds.), pp. 329–342, Proc. 1963 Symp. at Berkely, North-Holland, Amsterdam, 1965, 1970, 1972.

SEGERBERG, K. (1971), "An Easy in Classical Modal Logic," Filosofiska Studier, # 13, Uppsala.

SEGERBERG, K. (1973), Two-dimensional modal logic, *J. Philos. Logic* **2**, 77–96.

SEGERBERG, K. (1977), A completeness theorem in modal logic of programs, preliminary report, *Notices Amer. Math. Soc.* **24**, 6, A-552.

SHOENFIELD, J. (1967), "Mathematical Logic," Addison-Wesley, Reading.

SKORDEV, D. (1980), "Combinatory Spaces and Recursiveness in Them," Publ. House of the Bulg, Acad. of Sci., Sofia. [in Russian]

SKORDEV, D. (1984), A formal system for proving some properties of recursive programs in iterative combinatory spaces, *Fund. Inform.* **3**, 359–365.

SKORDEV, D., Ed. (1987), "Mathematical Logic and its Applications, Proc. of the Summer School and Conf. dedicated to the 80th anniversary of Kurt Gödel, Druzhba, 1986," Plenum, New York/London.

SOTIROV, V. (1984), "Intuitionistic Logic of Elementary Events," unpublished report, Sofia University.

SOTIROV, V. (1985), "Intuitionistic Logic with Dates," unpublished report, Sofia University.

STREETT, R. (1982), Propositional dynamic logic of looping and converse is elementary decidable, *Inform. and Control* **54**, 121–141.

SUNDHOLM, G. (1978), "On the ω-Rule," B.Ph. Thesis, Oxford.

TEHLIKELI, S. (1985), An alternative modal logic, internal semantics, and external syntax (A philosophical abstract of a mathematical essay), unpublished manuscript, Sofia.

THIELE, H. (1966), "Wissenschaftstheoretische Untersuchungen in algorithmischen Sprachen," Berlin.

TINCHEV, T. (1986), "Extensions of the Propositional Dynamic Logic," Ph.D. Thesis, Faculty of Mathematics, Sofia University. [In Bulgarian]

TINCHEV, T. (1988), On the expressibility of the PDL with local assignments, unpublished manuscript. [In Bulgarian]

TINCHEV, T., AND VAKARELOV, D. (1983), "Propositional Dynamic Logic with Recursive Programs," Banach Center Publications, to appear.

TINCHEV, T., AND VAKALEROV, D. (1985), Propositional dynamic logic with counters and stacks, *in* "Lecture Notes in Computer Science," Vol. 208, pp. 364–374, Springer-Verlag, Berlin/New York.

TIOMKIN, M., AND MAKOWSKI, J. (1985), Propositional dynamic logic with local assignments, *Theoret. Comput. Sci.* **36**, 71–87.

VAKARELOV, D. (1983), Filtration theorem for dynamic algebras with tests and inverse operator, *in* "Lecture Notes in Computer Science," Vol. 148, pp. 314–324, Springer-Verlag, Berlin/New York.

VAKARELOV, D. (1985–1988), private communication.

ZLOOF, M. (1976), "Quary-by-Example: Operations on the Transitive Closure," IBM Research Report RD 5526.